TOOLS IN APPLIED MACRO I

EXERCISE QUESTIONS

Basic time series processes

(1) Write down an MA(1) process. What is the variance of the process? What are the auto covariances?
(2) Write down an AR(1) process. What is the variance of the process? What are the autocovariances?
(3) Consider the first order difference equation

$$y_t = \phi y_{t-1} + w_t$$

Write y_t as a function of y_0 and $w_{t-s} : s = 0, 1, 2, 3, ...$
(4) What the is the dynamic multiplier, i.e. impulse response function of y_t to a one-off change in w_t? How about to a permanent change in w_t?
(5) Put the $ARMA(p, q)$ model

$$\begin{aligned} Y_t &= c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} \\ &\quad + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q} \end{aligned}$$

into a VAR(1) form.
(6) Under what conditions is an MA(1) process invertible?
(7) Show that there exists an invertible MA(1) process that is observationally equivalent to a non-invertible MA(1) process. What are the relationships between the parameters of the two processes?
(8) Explain a procedure for computing the covariance of the variables in a VAR(1). What are the auto covariances and how can they be found?
(9) Write down an ARMA(p,q) process as a function of y_t and ε_t process using lag operators.
(10) Derive the inverse of $(1 - \phi L)$. Under what conditions is this inverse well-defined?
(11) Rewrite the process

$$y_t = (1 - \phi_1 L - \phi_2 L^2)^{-1} w_t$$

as a VAR(1). Under what conditions is the resulting VAR(1) stable?
(12) Let $\lambda_1 = 1.5$ and $\lambda_2 = 0.2$ in

$$(1 - \lambda_1 L)(1 - \lambda_2 L) y_t = \varepsilon_t$$

Is the process stable? Rewrite it in the $ARMA(p, q)$ form. What is p and what is q? Find the corresponding coefficient as functions of λ_1 and λ_2.
(13) Define stationarity and ergodicity.

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Inner product spaces

(1) Consider the (Hilbert) space of square summable sequences l^2. Under what parameter conditions is
$$y_t = \rho y_{t-1} + u_t : u_t \sim N(0, \sigma^2)$$
an element of l^2 ?

(2) Find the inner products $\langle y_t, y_t \rangle$, $\langle y_t, y_{t-1} \rangle$ and $\langle y_t, x_t \rangle$ where
$$x_t = \phi x_{t-1} + u_t$$
and the shock u_t is the same as in the previous question.

(3) Use the projection theorem to find $E\left[x_t \mid x_{t-1}\right]$, $E\left[x_t \mid y_t\right]$ and $E\left[x_t \mid y^t\right]$ where $y^t = \{y_s\}_{s=-\infty}^{t}$.

(4) Define the metric $d(y, x)$ for $y, x \in l^2$ as
$$d(y, x) \equiv \sqrt{\|y - z\|}$$
where $\|\cdot\|$ is the norm on l^2. Find the distances $d(x_t, E\left[x_t \mid x_{t-1}\right])$, $d(x_t, E\left[x_t \mid y_t\right])$ and $E\left[x_t \mid y^t\right]$

Estimation

(1) Describe two different ways of estimating the parameters of a VAR.

(2) What is the exact likelihood function and what is the conditional likelihood?
   - (a) Under what conditions can you use the exact likelihood?
   - (b) What is the relationship between the exact likelihood function, the conditional likelihood function and OLS?

(3) Use the projection theorem to find the OLS estimates of the parameters of a VAR(2)

Numerical Maximization

(1) Consider the model
$$y_t = \rho_1 y_{-1} + \rho_2 y_{t-2} + u_t$$
   - (a) What is the vector θ of parameters?
   - (b) Describe a grid search procedure to find the MLE $\widehat{\theta}$
   - (c) What are the advantages and disadvantages of grid search?
   - (d) Describe the steepest ascent method to find the MLE $\widehat{\theta}$
   - (e) What are the advantages and disadvantages of steepest ascent?
   - (f) Describe a simulated annealing algorithm that can be used to find the MLE $\widehat{\theta}$
   - (g) What are the advantages and disadvantages of simulated annealing?

Structural VARS

(1) Why can't we estimate A_0 and A_1 in the structural form
$$A_0 \mathbf{y}_t = A_1 \mathbf{y}_{t-1} + \mathbf{u}_t$$
directly?

(2) Describe how to identify A_0 and A_1 using contemporaneous restrictions.

   (a) How many restrictions do you need if y_t is n dimensional?
(3) Describe how to identify A_0 and A_1 using long run restrictions.
   (a) How many restrictions do you need if y_t is n dimensional?
(4) Find the impulse response function for the structural form above.
(5) Describe a procedure to decompose the variance of y_t into the fractions caused by the individual shocks in the vector u_t
(6) Structural VARS: What's good about them?
(7) Structural VARS: What's not so good about them?

Principal components and factor models

(1) Describe a scree plot and how they can be used to determine the number of factors.
(2) (a) For the $(N \times T)$ dimensional data set Z^T with covariance Σ_z, describe how the first three principal components factors can be extracted.
   (b) Consider the state space system

$$\begin{aligned} X_t &= AX_{t-1} + C\mathbf{u}_t : \mathbf{u}_t \sim N(0, I) \\ Z_t &= DX_t + R\mathbf{v}_t : \mathbf{v}_t \sim N(0, I) \end{aligned}$$

   Show how to find the covariance $\Sigma_z \equiv E(Z_t Z_t')$ of the vector Z_t as a function of the matrices A, C, D and R.
   (c) Find the impulse response function for any variable in Z_t to any shock in $\mathbf{u}_t$ as a function of the matrices A, C, D and R.
   (d) Assume that Z^T in 2a) were generated by the state space system in 2b). Find a mapping between the principal component factors F_t and the state vector X_t.

State Space Models and the Kalman Filter

(1) Find the state space form

$$\begin{aligned} X_t &= AX_{t-1} + C\mathbf{u}_t \\ Z_t &= DX_t + \mathbf{v}_t \end{aligned}$$

of the VARMA(p,q)

$$\begin{aligned} y_t = &\ \Phi_1 y_{t-1} + ...\Phi_p y_{t-p} \\ &+\varepsilon_t + \Psi_1 \varepsilon_{t-1} + ... + \Psi_q \varepsilon_{t-q} \end{aligned}$$

where $Z_t = y_t$.
(2) For the scalar process

$$\begin{aligned} x_t &= \rho x_{t-1} + u_t \\ z_t &= x_t + v_t \\ \begin{bmatrix} u_t \\ v_t \end{bmatrix} &\sim N\left(0, \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}\right) \\ x_{0|0} &= \overline{x}_0 \\ E\left(\overline{x}_0 - x_0\right)^2 &= p_{0|0} \end{aligned}$$

find the Kalman gain k_t such that $x_{t|t}$ given by

$$x_{t|t} = \rho x_{t-1|t-1} + k_t \left[z_t - \rho x_{t-1|t-1}\right]$$

is the linear minimum variance estimate of x_t conditional on $\overline{x}_0$ and the history of z_t.

(3) What is k_t if $\sigma_v^2 = 0$? Interpret.
(4) What is k_t if $\sigma_v^2 = \infty$? Interpret.
(5) What is k_t if $\sigma_u^2 = 0$? Interpret.
(6) What is k_t if $\sigma_u^2 = \infty$? Interpret.
(7) Describe how k_t depends on the observations $z_t, z_{t-1}, ...z_1$
(8) What is the log likelihood function of the state space system

$$\begin{aligned} X_t &= AX_{t-1} + C\mathbf{u}_t \\ Z_t &= DX_t + \mathbf{v}_t \end{aligned}$$

?

(9) Consider the state space system of the form in Q8 above and define

$$P_{t|t-s} \equiv E\left[X_t - E\left(X_t \mid Z^{t-s}\right)\right]\left[X_t - E\left(X_t \mid Z^{t-s}\right)\right]'$$

(a) What restrictions on A, C, D and Σ_v would imply that
(b) $P_{t|t} = 0$?
(c) $P_{t|t-1} = CC'$?
(d) $P_{t|t-1} = E\left(X_t X_t'\right)$?
(e) What are the upper and lower bounds of $P_{t|t}$ and $P_{t|t-1}$?

(10) Find a rcursive expression for $P_{t|t-1}$.
(11) Find an expression for the Kalman gain K_t such that $X_{t|t}$ in

$$X_{t|t} = AX_{t-1|t-1} + K_t\left[Z_t - AX_{t-1|t-1}\right]$$

is the minimum variance estaimte of X_t.

(12) In what sense is the Kalman filter optimal if the Gaussian error assumption is relaxed?
(13) Orthogonolize the matrix

$$Y \equiv \left[\begin{array}{ccc} y_1 & \cdots & y_n \end{array}\right]$$

using the Gram-Schmidt procedure.