Tools for Applied Macro

Lecture 8: Determining lag order of a VAR(p)

February 27, 2017

Today:

Based on Lutkepohl Ch 4.1, 4.2, 4.3

- How to choose the VAR order

A VAR(p) model

VAR(p) process:

$$\underset{(n\times 1)}{\mathbf{y}_t} = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \ldots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \varepsilon_t$$

We now know how to find c and $\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots \mathbf{\Phi}_p, \mathbf{\Omega}$ for a given p.

- But how do we choose the "right" p?

Choosing p

Fundamental trade off:

In-sample-fit versus over-parameterization

- More lags always makes $\widehat{\Omega}$ smaller
- But more lags decreases precision of estimates of $\mathbf{\Phi}$
  - This is captured by the small sample "corrected" one step ahead forecast error covariance

$$\widehat{\Omega}(1) = \frac{T + np + 1}{T}\widehat{\Omega}$$

There are several ways to determine the appropriate p and the best choice depends on context

Likelihood ratio test statistic

Compares fit of model while penalizing models with a larger number of parameters

$$\begin{aligned} \lambda_{LR} &= 2\left(L_1 - L_0\right) \\ &= T\left[\log\left|\widehat{\Omega}_0\right| - \log\left|\widehat{\Omega}_1\right|\right] \end{aligned}$$

The log likelihood of unrestricted model cannot be lower than that of unrestricted model, i.e. $L_1 \geq L_0$

- Critical values determine how much larger unrestricted likelihood has to be in order to reject the restricted model
- Critical values are increasing in the number of restrictions

A procedure to choose the lag order p

Assume upper bound M of p is known:

1. $\mathrm{H}_0^1 : \mathbf{\Phi}_M = 0$ versus $\mathrm{H}_1^1 : \mathbf{\Phi}_M \neq 0$
2. $\mathrm{H}_0^2 : \mathbf{\Phi}_{M-1} = 0$ versus $\mathrm{H}_1^2 : \mathbf{\Phi}_{M-1} \neq 0 \mid \mathbf{\Phi}_M = 0$
3. $\vdots$
4. $\mathrm{H}_0^M : \mathbf{\Phi}_1 = 0$ versus $\mathrm{H}_1^M : \mathbf{\Phi}_M \neq 0 \mid \mathbf{\Phi}_M = \cdots = \mathbf{\Phi}_2 = 0$

One need to be careful with distinguishing significance level of individual test versus significance of over all procedure since we may falsely reject H_0^1 and therefore never test H_0^2,H_0^3, ...H_0^M

German investment/income/consumption model

Lutkepohl's example:

VAR order m	$\widehat{\Omega}(m) \times 10^4$	$\lvert\widehat{\Omega}(m)\rvert \times 10^{11}$
0	$\begin{bmatrix} 21.8 & .41 & 1.23 \\ \cdot & 1.42 & .57 \\ \cdot & \cdot & 1.01 \end{bmatrix}$	2.47
2	$\begin{bmatrix} 19.2 & .62 & 1.13 \\ \cdot & 1.27 & .57 \\ \cdot & \cdot & .82 \end{bmatrix}$	1.26
4	$\begin{bmatrix} 17.0 & .57 & 1.25 \\ \cdot & 1.23 & .54 \\ \cdot & \cdot & .77 \end{bmatrix}$	.96

LR statistics for invst/income/cons example

The LR test statistic is given by the log difference of the determinants of covariance matrices of the estimated residuals

$$\lambda_{LR}(i) = T\left[\log\left|\widehat{\Omega}(m-i-1)\right| - \log\left|\widehat{\Omega}(m-i)\right|\right]$$

i	H_0^i	m under H_0^i	λ_{LR}^a
1	$\mathbf{\Phi}_4 = 0$	3	14.44
2	$\mathbf{\Phi}_3 = 0$	2	4.76
3	$\mathbf{\Phi}_2 = 0$	1	24.90
4	$\mathbf{\Phi}_1 = 0$	0	23.25

[a]Critical value for individual 5% level test $\chi^2(9)_{.95} = 16.92$

This procedure thus suggest that we should choose $p = 2$.

Alternative criteria for choosing VAR order p

LR procedure above tries to estimate the "true" p

- But perhaps we do not really care about p?
  - Choose p that minimizes forecast MSE

Small versus large sample criteria

- Consistent order selection

Akaike's Final Prediction Error Criterion (FPE)

Choose p such that approximate 1-step ahead forecast MSE are minimized

$$\widehat{\Omega}(1) = \frac{T + np + 1}{T}\widehat{\Omega}$$

and use

$$\widehat{\Omega} = \frac{T}{T - np - 1}\tilde{\Omega}(p)$$

as the estimated error covariance. Taking the determinant of the combination of (1) and (2) and gives the FPE

$$FPE(p) = \left[\frac{T + np + 1}{T - np - 1}\right]^n \times \left|\tilde{\Omega}(p)\right|$$

Choose p that minimizes FPE

Akaike Information Criterion (AIC)

AIC is very similar to FPE though motivation is different:
Choose p to minimize

$$AIC(p) = \ln\left|\widehat{\Omega}(p)\right| + \frac{2pn^2}{T}$$

where pn^2 is the number of freely estimated parameters.

Consistent order selection

A order selection criterion is called “consistent” if asymptotically (i.e. for large T) it selects the true p with probability 1.

- FPE and AIC do not select true p with prob 1 but tend to over predict the number of lags needed, i.e. $\widehat{p} > p$ with $prob > 0$ as $T \to \infty$

Consistent alternatives:

- Hannan-Quinn

$$HQ(p) = \ln \left| \widehat{\Omega}(p) \right| + \frac{2 \ln \ln T}{T} pn^2$$

- Schwarz

$$SC(p) = \ln \left| \widehat{\Omega}(p) \right| + \frac{\ln T}{T} pn^2$$

A Matlab example

Housekeeping:

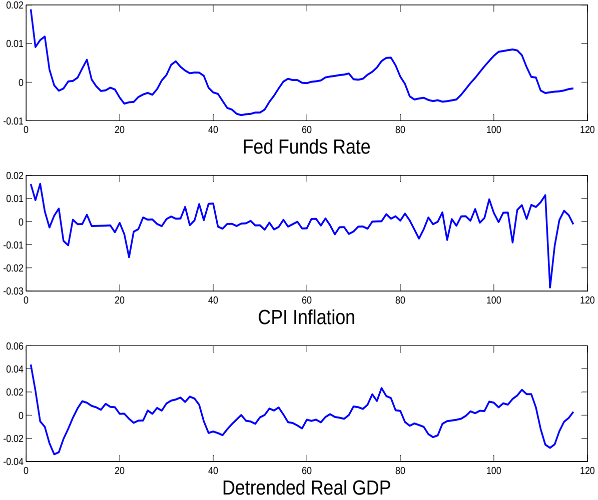
```
clear all;
clc;
load('Y'); T=length(Y);
```

A Matlab example

Have a look at the data

```
figure(1)
subplot(3,1,1);plot(Y(1,:),'linewidth',2)
xlabel('Fed Funds Rate','fontsize',20)
subplot(3,1,2);plot(Y(2,:),'linewidth',2)
xlabel('CPI Inflation','fontsize',20)
subplot(3,1,3);plot(Y(3,:),'linewidth',2)
xlabel('Detrended Real GDP','fontsize',20)
```

Have a look at the data



A Matlab example

Estimate VAR(p)

```
n=length(Y(:,1));
p=4;
y=Y(:,p+1:end);
Z=[ones(1,T-p);];

for j=1:p;
   Z=[Z; Y(:,p+1-j:end-j);];
end

Btilda=y*Z'/(Z*Z');
Sigmatilda=(1/(T-p))*(y-Btilda*Z)*(y-Btilda*Z)';
```

A Matlab example

Compute Criteria

FPE $=$([(T$+$n*p$+$1)/(T-n*p-1)]^n) $* \, det(Sigmatilda)$
AIC $=$ log(det(Sigmatilda))$+$ (2*p*n^2)/T
SC $=$ log(det(Sigmatilda))$+$ (log(T)*2*p*n^2)/T
HQ $=$ log(det(Sigmatilda))$+$ (log(log(T))*2*p*n^2)/T

A Matlab example

Set max lag order M etc

```
M=12;
DETSIGVEC=[];
FPE=[];
AIC=[];
SC=[];
HQ=[];
```

A Matlab example

Loop to compute criteria and LR statistics

```
for p=M:-1:0
   y=Y(:,p+1:end);
   Z=[ones(1,T-p);];
   for j=1:p;
      Z=[Z; Y(:,p+1-j:end-j);];
   end
   Btilda=y*Z'/(Z*Z');
   Sigmatilda=(1/(T-p))*(y-Btilda*Z)*(y-Btilda*Z)';
   DETSIGVEC=[DETSIGVEC log(det(Sigmatilda))];
   FPE=[ ([(T+n*p+1)/(T-n*p-1)]^n) * det(Sigmatilda)  FPE ];
   AIC=[ log(det(Sigmatilda))+ (2*p*n^2)/T  AIC ];
   SC=[ log(det(Sigmatilda))+ (log(T)*2*p*n^2)/T  SC ];
   HQ=[ log(det(Sigmatilda))+ (log(log(T))*2*p*n^2)/T  HQ ];
end
```

A Matlab example

Compute the LR test statistic

```
for j=1:M;
   DETSIGDIFF(j)=T*(DETSIGVEC(j+1)-DETSIGVEC(j));
end
```

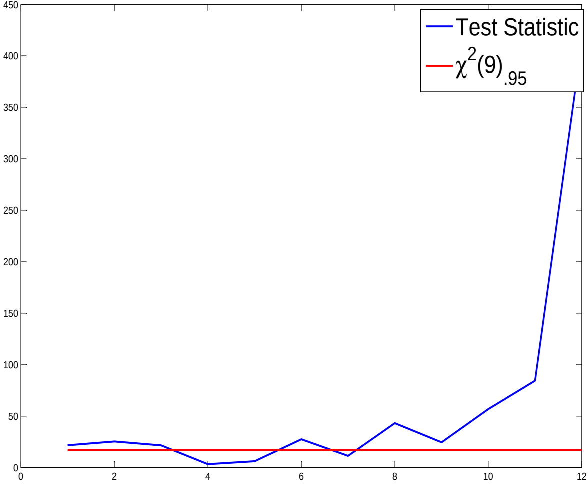
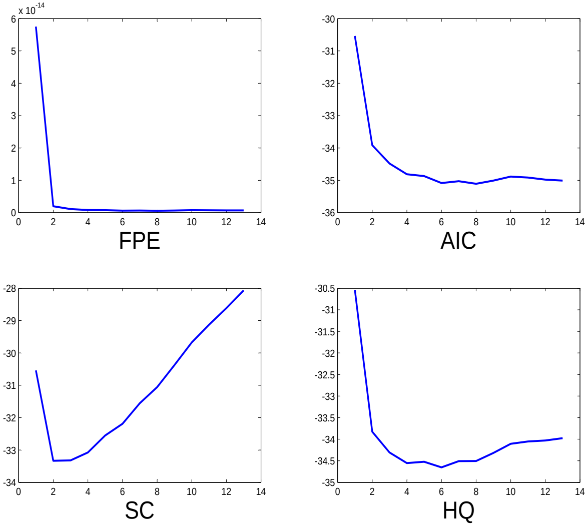


Figure: LR test statistic


x 10^-14
FPE
AIC
SC
HQ