PRIVATE AND PUBLIC INFORMATION

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Most economic models involve some type of interaction between multiple agents where the payoff of one agent depends not only on the actions taken by him, but also on the actions taken by other agents. When agents' preferences and environment are identical and all share the same information, an individual agent can infer the actions that others will take by introspection, since all agents will choose the same action in equilibrium. If agents have access to private information, this is no longer possible since individual agents cannot know with certainty what other agents know and therefore also not know with certainty what actions they will take. It then becomes necessary for agents to form expectations about the actions of others. Additionally, to predict the behavior of agents that form expectations about the actions of others, one need to form expectations about other agents' expectations about the actions of others, and so on, leading to the so-called infinite regress of expectations.[1] The idea that agents observe different pieces of information has a lot of appeal and has been applied to a variety of settings, including general equilibrium models of the business cycle and asset pricing models.[2] However, as a consequence of the infinite regress problem one could characterize most existing models of private information and strategic interaction as efforts to avoid modeling higher order expectations explicitly, and instead find alternative representations where higher order expectations do not occur as state variables. Notable exceptions are Woodford (2002), Morris and Shin (2002) and Adam (forthcoming) who by restricting their attention to models of static decisions are able to analyze higher order expectations explicitly.

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[1] Townsend (1983) and Sargent (1991).

[2] Some examples are Townsend (1983), Sargent (1991), Woodford (2002), Lorenzoni (2005), Bacchetta and Van Wincoop (2005), Kasa, Walker and Whiteman (2006) and Cespa and Vives (2007).

The paper by Morris and Shin (2002) is (together with Townsend 1983) one of the most often cited papers in this literature. In it, the authors show that agents tend to behave as if they put "too much" weight on public signals relative to private signals when there are strategic complementarities in actions, that is, when agent's utility is decreasing in the distance of their own action from others' actions. This makes intuitive sense: The public signal conveys information about the information available to others and naturally becomes more important when the actions of other agents matter for an individual's optimal decision. We will derive the solution and some of the results from Morris and Shin's paper below. Before analyzing the economics of private and public information though, it is necessary to invest some time in a notational machinery as well as to define exactly what is meant by a higher order expectation.

1. Higher order expectations: Concepts and notation

There is a continuum of agents indexed by j. Agent j's first order expectation of the variable θ_t is agent j's best estimate of the value of the variable given his information set Ω_t^j. We denote agent j's first order expectation of θ_t at time t

$$\theta_t^{(1)}(j) \equiv E\left[\theta_t \mid \Omega_t^j\right] \tag{1.1}$$

The average first order expectation is obtained by taking averages of (1.1) across agents

$$\theta_t^{(1)} \equiv \int E\left[\theta_t \mid \Omega_t^j\right] \quad dj \tag{1.2}$$

The average second order expectation is obtained by taking the average of agents' expectations of (1.2)

$$\theta_t^{(2)} \equiv \int E\left[\theta_{t|t}^{(1)} \mid \Omega_t^j\right] \ dj \tag{1.3}$$

The average contemporaneous second order expectation of θ_t thus is the average expectation at time t of the average expectation at time t of the value of θ_t. We can generalize this notation to the k^{th} order expectation of θ_t

$$\theta_t^{(k)} \equiv \int E\left[\theta_{t|t}^{(k-1)} \mid \Omega_t^j\right] \quad dj \tag{1.4}$$

Define the zero order expectation of θ_t as the actual value of the variable

$$\theta_t^{(0)} \equiv \theta_t \tag{1.5}$$

In general

$$\theta_t^{(k)} \neq \theta_t^{(k+l)} \tag{1.6}$$

for $l \neq 0$. We call a sequence of expectations, for instance from order zero to k, a *hierarchy* of expectations from order zero to k. Vectors consisting of a hierarchy of expectations are denoted

$$\theta_t^{(0:k)} = \left[\begin{array}{cccc} \theta_t^{(0)} & \theta_t^{(1)} & ... & \theta_t^{(k)} \end{array}\right]' \tag{1.7}$$

2. Rationality and expectations about others' expectations

In rational expectations models, (first order) expectations are pinned down by the structure of the model. That is, an agent's expectations should be the mathematical expectation of the variable in question, conditional on the information set available to the agent. The underlying assumption we make is thus that agents know the structure of the economy, that is, agents know the functional form and true parameter values of the model. Similarly, second order knowledge of rationality can be used to pin down second order expectations. That is, a rational agent's expectations can also be predicted, and treated as a random variable like any other. If an agent wants to form an expectation about another agents expectation, and knows that the other agent is rational, then second order expectation will be the rational expectation conditional on the expected information set of the other agent. A similar logic can be applied to third an higher order expectations.

2.1. **A simple example.** Consider the unobservable variable θ given by

$$\theta \sim N\left(0, \sigma_\theta^2\right) \tag{2.1}$$

Agents (indexed by j) observe a private noisy signal of θ given by

$$\begin{aligned} z(j) &= \theta + \eta(j) \\ \eta(j) &\sim N\left(0, \sigma_\eta^2\right) \forall j \end{aligned} \tag{2.2}$$

That is, all agents receive an equally precise signal of θ but agent j only observes his own signal $z(j)$. The optimal estimate of θ conditional on $z(j)$ is then given by

$$E\left[\theta \mid z(j)\right] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\eta^2} z(j) \tag{2.3}$$

$$= gz(j) \tag{2.4}$$

To find the average first order expectation, we just take averages, that is integrate over j to get

$$\begin{aligned} \theta^{(1)} &= \int E\left[\theta \mid z(j)\right] \quad dj \\ &= g\theta + g \int \eta(j)\ dj \\ &= g\theta \end{aligned} \tag{2.5}$$

where the last equality follows from the fact that the idiosyncratic noise terms average out to zero, i.e. $\int \eta(j)\ dj = 0$. The average first order expectation is thus a linear function of the true state θ. Agent j's second order expectation, that is agent j's expectation of the average expectation of θ is then given by

$$E\left[\theta^{(1)} \mid z(j)\right] = gE\left[\theta \mid z(j)\right] \tag{2.6}$$

$$= ggz(s) \tag{2.7}$$

Again taking averages across agents gives us the average second order expectation of θ

$$\theta^{(2)} = \int E\left[\theta^{(1)} \mid z(j)\right] \quad dj \tag{2.8}$$

$$= g^2\theta + g^2 \int \eta(j)\ dj \tag{2.9}$$

$$= g^2\theta \tag{2.10}$$

We could continue this indefinitely using that the average k^{th} order expectation will be given by

$$\theta^{(k)} = g^k\theta \tag{2.11}$$

3. Public Information and Strategic Interaction

In an influential paper in the AER from 2002, Stephen Morris and Hyun Shin demonstrated that in the combination of strategic complementarity and private information can make the impact of public signals disproportionately large and markets can appear to "overreact" to news. This result is derived in a setting with fully rational agents. Whenever anyone writes a paper about this topic, it is customary to refer to Keynes' "beauty contest" metaphor of stock markets. So here it is:

"It is not a case of choosing those [faces] which, to the best of one's judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees." Keynes, General Theory of Employment Interest and Money, 1936.

3.1. Morris and Shin's model.

Utility of agent i is given by

$$U_i = -\left(1-r\right)\left(a_i - \theta\right)^2 - r\left(L_i - \overline{L}\right) \tag{3.1}$$

where a_i is the action taken by agent i and

$$L_i = \int (a_j - a_i)^2 \, dj \tag{3.2}$$

and

$$\overline{L} = \int L_j dj \tag{3.3}$$

Agent i's first order condition is

$$a_i = (1 - r) E [\theta \mid I(i)] + rE [\overline{a} \mid I(i)] \tag{3.4}$$

where $\overline{a}$ is the average action, i.e. $\int a_i di$. Agents observe two signals of θ. The public signal y

$$\begin{aligned} y &= \theta + \eta \\ \eta &\sim N(0, \sigma_\eta^2) \end{aligned} \tag{3.5}$$

and the private signal x_i

$$\begin{aligned} x_i &= \theta + \varepsilon_i \\ \varepsilon_i &\sim N(0, \sigma_\varepsilon^2) \forall i \end{aligned} \tag{3.6}$$

The conditional first order expectation of θ will then be

$$E [\theta \mid x_i, y] = \frac{\sigma_\eta^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} (x_i - y) + y \tag{3.7}$$

$$= g (x_i - y) + y \tag{3.8}$$

and the general expression for a k^{th} order expectation

$$\theta^{(k)} = g^k (\theta - y) + y \tag{3.9}$$

3.2. **Equilibrium.** Conjecture a solution of the form

$$a_i = \kappa x_i + (1-\kappa)\,y \tag{3.10}$$

Substitute into FOC to get

$$\kappa x_i + (1-\kappa)\,y = (1-r)\left[g\,(x_i - y) + y\right] + r\left[\kappa\,(g\,(x_i - y) + y) + (1-\kappa)y\right] \tag{3.11}$$

equate coefficients on x_i

$$\kappa = (1-r)\,g + r\kappa g \tag{3.12}$$

$$= \frac{(1-r)\,g}{1-rg} \tag{3.13}$$

We can check that the limits makes sense:

- When σ_η^2 (and g) tends to zero, κ tends to zero: When the public signal is perfectly accurate, the optimal action put zero weight on the private signal.
- When σ_η^2 tends to infinity (and g tends to 1), κ tends to one: When the public signal is infinitely noisy, the optimal action put zero weight on the public signal.
- When $0 < g < 1$ and r decreases and tends to 0 (i.e. decreasing utility from coordination), κ increases from it's minimum zero (at $r = 1$) towards it maximum g (at $r = 0$) which also makes sense: When no weight is put on coordination ($r = 0$), agents will simply choose to minimize the distance between their action and the fundamental θ, i.e. $r = 0 \implies a_i = E\left[\theta \mid x_i, y\right] = gx_1 + (1-g)y$.

3.3. **An alternative solution method.** Instead of using the method of undetermined coefficients, we can use a method that explicitly phrases the the average action $\overline{a}$ as a function of higher order expectations of θ. Start by taking averages of (3.4) to get

$$\overline{a} = (1-r)\,\theta^{(1)} + r\overline{a}^{(1)} \tag{3.14}$$

where

$$\overline{a}^{(1)} \equiv \int E\left[\overline{a} \mid I(i)\right] di \tag{3.15}$$

and then note that

$$\overline{a}^{(1)} = (1-r)\,\theta^{(2)} + r\overline{a}^{(2)} \tag{3.16}$$

or more generally

$$\overline{a}^{(k)} = (1-r)\,\theta^{(k+1)} + r\overline{a}^{(k+1)} \tag{3.17}$$

Repeated substitution of (3.17) into (3.14) gives the convergent sum

$$\overline{a} = (1-r)\sum_{k=1}^{\infty} r^{k-1}\theta^{(k)} \tag{3.18}$$

Now use that

$$\theta^{(k)} = g^k\,(\theta - y) + y \tag{3.19}$$

to get

$$\begin{aligned} \overline{a} &= (1-r)\sum_{k=1}^{\infty} r^{k-1}\left(g^k\,(\theta-y)+y\right) && (3.20)\\ &= \frac{(1-r)\,g}{1-rg}\theta + \frac{1-r}{1-r}y - \frac{(1-r)\,g}{1-rg}y && (3.21)\\ &= \frac{(1-r)\,g}{1-rg}\theta + \left(1-\frac{(1-r)\,g}{1-rg}\right)y && (3.22) \end{aligned}$$

which is the same solution as (3.13).

3.4. **Welfare.** Morris and Shin specifies the following social welfare function

$$\begin{aligned} W &\equiv \frac{1}{1-r}\int u_i di && (3.23)\\ &= -\int (a_i-\theta)^2\,di && (3.24) \end{aligned}$$

That is, social welfare is a function solely of the average squared distance between actions and fundamentals. We can substitute in our optimal strategy to get

$$\begin{aligned} E[W \mid \theta] &= -\int (\kappa x_i + (1-\kappa)y - \theta)^2 \, di && (3.25) \\ &= -\int [\kappa(\theta+\varepsilon_i) + (1-\kappa)(\theta+\eta) - \theta]^2 \, di && (3.26) \end{aligned}$$

Simplify

$$\begin{aligned} E[W \mid \theta] &= -\int [\kappa\varepsilon_i + (1-\kappa)\eta]^2 && (3.27) \\ &= -\kappa^2\sigma_\varepsilon^2 - (1-\kappa)^2\sigma_\eta^2 \end{aligned}$$

Morris and Shin then shows that

$$\frac{\partial E[W \mid \theta]}{\partial \sigma_\eta^2} \leq 0 \qquad (3.28)$$

if and only if

$$\frac{\sigma_\eta^2}{\sigma_\varepsilon^2} \leq \frac{1}{(2r-1)(1-r)} \qquad (3.29)$$

That is, more noise in the public signal can be good for welfare if the variance of the noise in the public signal is sufficiently large so that the inequality (3.29) is not satisfied.

3.4.1. *The Svensson critique.* Svensson (2006) argues that the inequality (3.29) is unlikely to hold, for two reasons. First, r must be larger than 0.5 for the right hand side to be positive (and the l.h.s. is always positive.) Secondly, $(2r-1)(1-r)$ has a maximum at $r = 3/4$ so the right hand side has minimum at $r = 3/4$ equal to 8, i.e. if

$$\frac{\sigma_\eta^2}{\sigma_\varepsilon^2} \leq 8 \qquad (3.30)$$

holds we always have that

$$\frac{\partial E[W \mid \theta]}{\partial \sigma_\eta^2} \leq 0. \qquad (3.31)$$

Svensson continues by arguing that it is unlikely that any public information is eight times less precise than private information and that most likely, more precise public information is beneficial for welfare.

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