Information, Learning and Expectations in Macro

Dynamic Higher Order Expectations

March 7, 2016

Private/heterogenous/dispersed information in dynamic models

Today:

- *Forecasting the Forecasts of Others* (Townsend JPE 1983)
- *Dynamic Higher Order Expectations* (Nimark WP 2011)

Private/heterogenous/dispersed information

Every agent has his own “window to the world” (Morris and Shin (2002)

- Old idea: Pigou (1929) argued that business cycles was caused by entrepreneurs in a given sector being imperfectly informed about other sectors in the economy
- This could give rise to mutually re-enforcing forecasts errors across industries, leading to ”waves of optimism and pessimism”

Private information is one way of introducing disagreement among agents and uncertainty about the plans and actions of other agents

Private/heterogenous/dispersed information

Is private information important?

- We need to solve quantitative models to answer that question
- The principal modeling difficulty: The infinite regress of "forecasting the forecasts of others" (Townsend 1983)

Solution strategies for models with private information

Most strategies rely on making private information "short-lived"

- Lagged revelation of shocks: Townsend (1983), Singleton (1987)
  - Not always realistic and can result in weird (kinked) IRF (Bacchetta and Van Wincoop (2006))
- Finite horizon: Allen, Morris and Shin (2006), Banerjee, Kaniel and Kremer (2009)
- Static choices: Woodford (2002), Morris and Shin (2002), Angeletos and La'o (2009)

How can we solve models when we relax these assumptions?

The infinite regress of expectations: It's your choice!

Remember: We could solve the Morris and Shin (2002) model with our without an explicit expression for higher order expectations: Either predict what you really care about

$$\begin{aligned} a_i &= (1-r)\,E\left[\theta \mid I(i)\right] + rE\left[\bar{a} \mid I(i)\right] \\ a_i &= \kappa x_i + (1-\kappa)y \end{aligned}$$

Or write an equivalent expression with an explicit role for higher order expectations

$$\begin{aligned} a_i &= (1-r)\sum_{k=1}^{\infty} r^{k-1}\theta^{(k)}(i) \\ \theta^{(k)} &= g^k(\theta - y) + y \end{aligned}$$

We will discuss both approaches in a dynamic setting today

Forecasting the forecasts of others
Townsend, (JPE 1983)

Some history:

- Townsend's paper the first to recognize the infinite regress of expectations in a macro setting, though it was already well-known within game theory.
- Solution method:
  - Make assumption about revelation of state to make relevant history finite.
- The model itself is not so interesting.

But we will still talk about the model of Townsend.

Forecasting the forecasts of others
Townsend, (JPE 1983)

Output sector i is function only of capital

$$y_t^i = f_0 k_t^i$$

Market clearing price

$$\begin{aligned} P_t^i &= -b_1 Y_t^i + z_t^i \\ z_t &= \theta_t + \epsilon_t^i \end{aligned}$$

where z_t is a demand shock with persistent component θ_t

$$\theta_t = \rho\theta_{t-1} + v_t$$

and $\epsilon_t^i \sim N(0, \sigma_\epsilon^2)$ and $v_t \sim N(0, \sigma_v^2)$

Forecasting the forecasts of others
Townsend, (JPE 1983)

The firm's profit max problem:

$$\max_{\{k_t^i\}_{t=1}^{\infty}} E_0^i \sum_{t=0}^{\infty} \beta^t \left[P_t^i f_0 k_t^i - \frac{f_1}{2} \left(k_t^i\right)^2 - \frac{f_2}{2} \left(k_{t+1}^i - k_t^i\right)^2 \right]$$

$$f_0, f_2 \quad > \quad 0, \qquad f_1 \geq 0$$

Forecasting the forecasts of others
Townsend, (JPE 1983)

Maximizing profits lead to the decision rule

$$k_{t+1} = \lambda_1 k_t^i + \frac{f_0 \beta \lambda_1}{f_2} \sum_{j=0}^{\infty} (\beta \lambda_1)^j E\left(P_{t+1+j}^i \mid \Omega_t^i\right)$$

which implies a law of motion for aggregate industry i capital stock

$$K_{t+1}^i = h_1 K_t^i + h_2 M_t^i$$

where

$$M_t^i = E\left(\theta_t \mid \Omega_t^i\right)$$

Case 1: Hierarchical information structure

Industry 1's information set is nested in the information set of Industry 2.

- Industry 1 is "self contained":
  - Does not observe prices in any other industry
- Industry 2 observes prices in Industry 1, but there is no trade or other "real" interaction across sectors
  - Industry 2 uses observation of price in Industry 1 to form an estimate of θ_t

The only link between industries is that they both try to estimate the same unobservable state

Industry 1 filtering problem and state law of motion

By Kalman filtering

$$M_t^1 = \rho M_{t-1}^1 + G_t \left(z_t^1 - \rho M_{t-1}^1 \right)$$

or with Townsend's notation:

$$M_t^1 = \alpha_0 M_{t-1}^1 + \alpha_1 z_t^1$$

This implies that law of motion of state of Industry 1 is given by

$$\begin{bmatrix} K_{t+1}^1 \\ M_{t+1}^1 \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & 0 \\ 0 & \alpha_0 & \alpha_1 \rho \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} K_t^1 \\ M_t^1 \\ \theta_t \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_1 v_{t+1} + \alpha_1 \epsilon_{t+1}^1 \\ v_{t+1} \end{bmatrix}$$

Given the decision rule, industry 1 is solved.

Industry 2 filtering problem

Firms in Industry 2 also want to estimate the current value of θ_t

- Industry 2 observe prices in both industries

Industry 2's information set is

$$\Omega_t^2 = \left\{ z_s^2, P_s^1, P_s^2, M_s^2, K_s^2 : s = 0, 1, 2, ..., t \right\}$$

Industry 2 filtering problem

The filtering problem of Industry 2 is a standard state space system:

State equation

$$\begin{bmatrix} K^1_{t+1} \\ M^1_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & 0 \\ 0 & \alpha_0 & \alpha_1 \rho \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_1 v_{t+1} + \alpha_1 \epsilon^1_{t+1} \\ v_{t+1} \end{bmatrix}$$

Measurement equation

$$\begin{bmatrix} z^2_t \\ P^1_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -b_1 f_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} \epsilon^2_t \\ \epsilon^1_t \end{bmatrix}$$

Industry 2 does not intrinsically care about K^1_t and M^1_t but need to control for their effect on the observables.

The law of motion of the system

The complete state vector of the model then evolves according to

$$\begin{bmatrix} K^1_{t+1} \\ M^1_{t+1} \\ \theta_{t+1} \\ K^2_{t+1} \\ M^2_{t+1} \\ E\left[K^1_{t+1} \mid \Omega^2_{t+1}\right] \\ E\left[M^1_{t+1} \mid \Omega^2_{t+1}\right] \end{bmatrix} = A \begin{bmatrix} K^1_{t} \\ M^1_{t} \\ \theta_{t} \\ K^2_{t} \\ M^2_{t} \\ E\left[K^1_{t} \mid \Omega^2_{t}\right] \\ E\left[M^1_{t} \mid \Omega^2_{t}\right] \end{bmatrix} + C \begin{bmatrix} v_t \\ \epsilon^1_t \\ \epsilon^2_t \end{bmatrix}$$

i.e. we have a finite state representation.

The mechanics of a hierarchical information structures

What breaks the infinite regress of expectations?

- Nested information sets and a property of projections
  - Information set of Industry 1 nested in that of Industry 2 so that $\Omega_t^1 \subseteq \Omega_t^2$
  - We know from the properties of projections that $\mathcal{P}_1\mathcal{P}_2X = \mathcal{P}_1X$ and $\mathcal{P}_2\mathcal{P}_1\mathcal{P}_2X = \mathcal{P}_1X$ if $\Omega^1 \subseteq \Omega^2$

Intuition: You cannot predict the error of a strictly better informed agent so $\mathcal{P}_1\left(X - \mathcal{P}_2X\right) = 0 \implies \mathcal{P}_1X = \mathcal{P}_1\mathcal{P}_2X$

Case 2: Firms in industry 1 can also observe prices in Industry 2

Then we need to include industry 1's expectations about industry 2 expectations about .....industry 1's expectations of θ_t... and so on

- This is the infinite regress of expectations problem: Natural state representations tend to become infinite.

Townsend suggested that we

- assume that shocks are observed perfectly with a finite lag and
- use (non-recursive) projection methods to find expectations

This works since projections on a finite dimensional observations vector now spans the same space as the entire history of observables so that we can compute

$$E[\theta_t | z_t^i, z_{t-1}^i, P_t^j, P_{t-1}^j, \theta_{t-2}]$$

Two reasons Townsend's model is special

Bugs or features?

- No real strategic interaction: Markets are only linked informationally
- In fact, there is no private information in equilibrium as shown by Sargent (JEDC 1991)

You'll be the judge.

Dynamic Higher Order Expectations Nimark (2011)

Dynamic Higher Order Expectations Nimark (2011)

Solving dynamic models with private information

- Recursive formulation with an explicit role for higher order expectations

Impose common knowledge of rationality

- By itself does not solve the "infinite regress problem" but makes thinking about higher order expectations tractable
- Show that impact of expectations diminishes "fast enough" as order increases which allows for an arbitrarily good approximation
- Use Singleton's (1987) model of asset pricing with disparately informed traders as vehicle for the argument

Notation

Agents are indexed by $j \in (0,1)$

$$
\begin{aligned}
\theta_{t|t}^{(0)} &\equiv \theta_t \\
\theta_{t|t}^{(1)} &\equiv \int E\left[\theta_{t|t}^{(0)} \mid \Omega_t(j)\right] \quad dj \\
\theta_{t|t}^{(2)} &\equiv \int E\left[\theta_{t|t}^{(1)} \mid \Omega_t(j)\right] \quad dj \\
\theta_{t|t}^{(k)} &\equiv \int E\left[\theta_{t|t}^{(k-1)} \mid \Omega_t(j)\right] \quad dj
\end{aligned}
$$

Dynamic Notation

$$
\begin{aligned}
\theta^{(1)}_{t+1|t} &\equiv \int E\left[\theta_{t+1} \mid \Omega_t(j)\right] \;\; dj \\
\theta^{(2)}_{t+2|t+1|t} &\equiv \int E\left[\theta^{(1)}_{t+2|t+1} \mid \Omega_t(j)\right] \;\; dj \\
\theta^{(k)}_{t+k|\cdots|t} &\equiv \int E\left[\theta^{(k-1)}_{t+k|\cdots|t+1} \mid \Omega_t(j)\right] \; dj
\end{aligned}
$$

Notation cont.

Denote a vector consisting of a *hierarchy of expectations* (from order zero to k)

$$\theta_{t|t}^{(0:k)} = \begin{bmatrix} \theta_{t|t}^{(0)} \\ \theta_{t|t}^{(1)} \\ \vdots \\ \theta_{t|t}^{(k)} \end{bmatrix}$$

Constructing a law of motion for expectations using common knowledge of rationality

- Illustrate how common knowledge of rationality impose structure on higher order expectations
- A simple example (no economics yet)

Estimating an unobservable process

The true process is an AR(1)

$$\theta_t = \rho\theta_{t-1} + v_t$$

In period t agent j observes the private noisy signal $s_t(j)$

$$\begin{aligned} s_t(j) &= \theta_t + \eta_t(j), \\ \eta_t(j) &\sim N\left(0, \sigma_\eta^2\right) \ \forall \ j \end{aligned}$$

Updating equation

$$\theta_{t|t}^{(1)}(j) = (1 - g_1)\,\rho\theta_{t-1|t-1}^{(1)}(j) + g_1 s_t(j)$$

Higher order estimates

A new state space system

$$\begin{bmatrix} \theta_t \\ \theta_{t|t}^{(1)} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ g_1\rho & (1-g_1)\rho \end{bmatrix} \begin{bmatrix} \theta_{t-1} \\ \theta_{t-1|t-1}^{(1)} \end{bmatrix} + \begin{bmatrix} 1 \\ g_1 \end{bmatrix} v_t$$

$$s_t(j) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_t \\ \theta_{t|t}^{(1)} \end{bmatrix} + \eta_t(j)$$

Higher order estimates

Agent j can now form expectations of actual and average first order expectations using the new updating equation

$$\begin{bmatrix} \theta_{t|t}^{(1)}(j) \\ \theta_{t|t}^{(2)}(j) \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ g_1\rho & (1-g_1)\rho \end{bmatrix} \begin{bmatrix} \theta_{t-1|t-1}^{(1)}(j) \\ \theta_{t-1|t-1}^{(2)}(j) \end{bmatrix}$$

$$+ \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}' \left(\begin{bmatrix} \theta_t \\ \theta_{t|t}^{(1)} \end{bmatrix} - \begin{bmatrix} \theta_{t-1|t-1}^{(1)}(j) \\ \theta_{t-1|t-1}^{(2)}(j) \end{bmatrix} \right) + \eta_t(j) \right)$$

Higher order estimates

Again taking averages gives a law of motion for $\theta_{t|t}^{(0:2)}$

$$\begin{bmatrix} \theta_{t|t}^{(0)} \\ \theta_{t|t}^{(1)} \\ \theta_{t|t}^{(2)} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ g_1\rho & (1-g_1)\,\rho & 0 \\ g_2\rho & (g_1-g_2)\rho & (1-g_1)\,\rho \end{bmatrix} \begin{bmatrix} \theta_{t-1|t-1}^{(0)} \\ \theta_{t-1|t-1}^{(1)} \\ \theta_{t-1|t-1}^{(2)} \end{bmatrix} + \begin{bmatrix} 1 \\ g_1 \\ g_2 \end{bmatrix} v_t$$

An equilibrium property of higher order expectations

Variance of higher order expectations is bounded by variance of lower order expectations

- Implied by common knowledge of rationality
- Important for approximation result

Bounded variance of higher order expectations

Lemma: The variance of trader j's expectation of θ_t is bounded by the variance of θ_t, i.e.

$$E\left[\theta_t\right]^2 \geq E\left[\theta_t^{(1)}(j)\right]^2$$

Proof: Define trader j's first order expectation error $\varepsilon_t^{(1)}(j)$ as

$$\theta_t \equiv \theta_t^{(1)}(j) + \varepsilon_t^{(1)}(j)$$

The error $\varepsilon_t^{(1)}(j)$ is orthogonal to $\theta_t^{(1)}(j) \in \Omega_t(j)$ so we have

$$E\left[\theta_t\right]^2 = E\left[\theta_t^{(1)}(j)\right]^2 + E\left[\varepsilon_t^{(1)}(j)\right]^2$$

The proof then follows from the fact that variances are non-negative so that

$$E\left[\theta_t\right]^2 \geq E\left[\theta_t^{(1)}(j)\right]^2$$

Bounded variance of higher order expectations

Lemma: The variance of the average expectation of θ_t is bounded by the variance of θ_t, i.e.

$$E\left[\theta_t\right]^2 \geq E\left[\theta_t^{(1)}\right]^2 \tag{1}$$

Proof: Trader j's first order expectations have an MA representation with variance

$$E\left[\theta_t^{(1)}(j)\right]^2 = E\left[A(L)v_t\right]^2 + E\left[B(L)\epsilon_t\right]^2 + E\left[C(L)\eta_t(j)\right]^2 \tag{2}$$

Since $\int \eta_t(j)dj = 0 \forall t$ the average first order expectation is simply

$$\begin{aligned} \theta_t^{(1)} &= A(L)v_t + B(L)\epsilon_t + \int C(L)\eta_t(j)dj && (3) \\ &= A(L)v_t + B(L)\epsilon_t && (4) \end{aligned}$$

with variance

$$E\left[\theta_t^{(1)}\right]^2 = E\left[A(L)v_t\right]^2 + E\left[B(L)\epsilon_t\right]^2 < E\left[\theta_t^{(1)}(j)\right]^2 \tag{5}$$

Bounded variance of higher order expectations

Proposition: The variance of higher order expectations of θ_t are bounded by the variance of lower order expectations, i.e.

$$E\left[\theta_t^{(k)}\right]^2 \geq E\left[\theta_t^{(k+1)}\right]^2$$

Proof: To prove the proposition, replace the definition of trader j's first order expectations error $\varepsilon_t^{(1)}(j)$ in the proof of Lemma 1 with the definition of the k order expectation error

$$\theta_t^{(k-1)} - \theta_t^{(k)}(j) \equiv \varepsilon_t^{(k)}(j)$$

Noting that the k order error $\varepsilon_t^{(k)}(j)$ is orthogonal to $\theta_t^{(k)}(j) \in \Omega_t(j)$ allows for recursively establishing the proposition for $k = 2, 3, ...$by following the same steps as in the proofs of Lemma 1 and 2.

Bounded variance of higher order expectations

Proposition: The variance of higher order expectations of future expectations of θ_t are bounded by the variance of lower order expectations, i.e.

$$E\left[\theta^{(k-1)}_{t+k|\cdots|t}\right]^2 \geq E\left[\theta^{(k)}_{t+k+1|\cdots|t}\right]^2$$

Proof: To prove the proposition, replace the definition of trader j's first order expectations error $\varepsilon^{(1)}_t(j)$ in the proof of Lemma 1 with the definition of the k order future expectation error

$$\theta^{(k-1)}_{t+k|\cdots|t} - \theta^{(k)}_{t+k+1|\cdots|t}(j) \equiv \varepsilon^{(k+1)}_{t+k+1|\cdots|t}(j)$$

Again, since the k order error $\varepsilon^{(k+1)}_{t+k+1|\cdots|t}(j)$ is orthogonal to $\theta^{(k)}_{t+k+1|\cdots|t}(j) \in \Omega_t(j)$, the same recursive procedure as in Proposition 1 can be applied to establish the desired result for $k = 1, 2, 3, ...$

The Singleton (1987) asset pricing model

The Singleton (1987) asset pricing model

Continuum of traders indexed by j with CARA utility

Trader j's demand

$$z_t^d(j) = \frac{\left(E\left[p_{t+1} \mid \Omega_t(j)\right] - (1+\bar{r})\, p_t\right)}{\gamma\delta}$$

Supply

$$\begin{aligned} z_t^s &= \xi p_t + \theta_t + \epsilon_t \\ \theta_t &= \rho\theta_{t-1} + v_t \end{aligned}$$

The Singleton asset pricing model

Equilibrium price

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid \Omega_t(j)\right] \quad dj \right) - \delta\gamma\lambda\left[\theta_t + \epsilon_t\right]$$

where

$$0 < \lambda < 1$$

The Singleton asset pricing model

Trader j's information set

$$\begin{aligned} \Omega_t(j) &= \{s_{t-T}(j), p_{t-T} : T \geq 0\} \\ s_t(j) &= \theta_t + \eta_t(j) \end{aligned}$$

Singleton:

$$I_t^S = \{s_{t-T}(j), p_{t-T} : T \geq 0; v_{t-T}, \epsilon_{t-T} : T \geq 2\}$$

Singleton's Solution method

Same as Townsend's:

- Assume that everything is revealed with a two period lag:

$$\begin{aligned} p_t &= \lambda \left(\int E\left[p_{t+1} \mid s_t(j), s_{t-1}(j), p_t, p_{t-1}, \theta_{t-2}\right] \; dj \right) \\ &\quad -\delta\gamma\lambda\left[\theta_t + \epsilon_t\right] \end{aligned}$$

This allows for a finite and non-expanding dimension of state.

We will show how to solve the model without making this assumption.

The Price Euler Equation

The key structural equation is the price Euler equation

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid \Omega_t(j)\right] \quad dj \right) - \delta\gamma\lambda\left[\theta_t + \epsilon_t\right]$$

where

$$0 < \lambda < 1$$

The Full Information Equilibrium Price

Iterate price Euler-equation

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid \Omega_t(j)\right] \quad dj \right) - \delta\gamma\lambda\left[\theta_t + \epsilon_t\right]$$

forward

$$\begin{aligned} p_t \quad = \quad & -\delta\gamma\lambda\left(\theta_t + \epsilon_t\right) + \\ & -(\delta\gamma\lambda)\lambda\rho\theta_t + \ldots \\ & \ldots - (\delta\gamma\lambda)\left(\lambda\rho\right)^{\infty}\theta_{t+\infty} \end{aligned}$$

to get

$$p_t = -\frac{\delta\gamma\lambda}{1-\lambda\rho}\theta_t - \delta\gamma\lambda\epsilon_t$$

The Private Information Equilibrium Price: Complications

Iterate price Euler-equation forward

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid \Omega_t(j)\right] \quad dj \right) - \delta\gamma\lambda\left[\theta_t + \epsilon_t\right]$$

to get price as a function of higher order expectations of future θ

$$\begin{aligned} p_t \quad = \quad & -\delta\gamma\lambda\left(\theta_t + \epsilon_t\right) - (\delta\gamma\lambda)\lambda \int E\left[\theta_{t+1} \mid \Omega_t(j)\right] dj \\ & +(\delta\gamma\lambda)\lambda^2 \int E\left[\int E\left[p_{t+2} \mid I_{t+1}(j)\right] dj \mid \Omega_t(j) \right] dj] \end{aligned}$$

or

$$p_t = -\delta\gamma\lambda\epsilon_t - \delta\gamma\lambda \sum_{k=0}^{\infty} \lambda^k \theta^{(k)}_{t+k|\cdots|t}$$

An average expectations operator

Conjecture a law of motion for hierarchy

$$\theta_{t|t}^{(0:\infty)} = M\theta_{t-1|t-1}^{(0:\infty)} + N \begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix}$$

Define new operator $H : \mathbb{R}^\infty \to \mathbb{R}^\infty$

$$H \equiv \begin{bmatrix} \mathbf{0}_{\infty\times 1} & I \end{bmatrix}$$

then higher order expectations of future fundamental are given by

$$\int E\left[\theta_{t+1} \mid \Omega_t(j)\right] dj = e_1' MH\theta_{t|t}^{(0:\infty)}$$

$$\int E\left[\int E\left[\theta_{t+2} \mid I_{t+1}(j)\right] dj \mid \Omega_t(j)\right] dj] = e_1'(MH)^2\theta_{t|t}^{(0:\infty)}$$

...and so on.

The price function

The price of the asset can then be written as a function of the state

$$p_t = \begin{bmatrix} a_0 & a_1 & \cdots & a_\infty \end{bmatrix} \begin{bmatrix} \theta_{t|t}^{(0)} \\ \theta_{t|t}^{(1)} \\ \vdots \\ \theta_{t|t}^{(\infty)} \end{bmatrix} - \delta\gamma\lambda\epsilon_t$$

where the vector $\mathbf{a}$ resembles a discounted geometric sum of expected future fundamentals

$$\mathbf{a} \equiv \begin{bmatrix} a_0 & a_1 & \cdots & a_\infty \end{bmatrix}$$

A finite dimensional approximation

A finite dimensional approximation

Consider only the first $\overline{k}$ orders of expectations

$$p_{\overline{k},t} = -\delta\gamma\lambda\epsilon_t - \delta\gamma\lambda\sum_{k=0}^{\overline{k}}\lambda^k\theta^{(k)}_{t+k|\cdots|t}$$

We then have that

$$p_{\overline{k},t} = -\delta\gamma\lambda\epsilon_t - \delta\gamma\lambda\sum_{k=0}^{\overline{k}} e_1'\left(\lambda MH\right)^k\theta_t^{(0:\overline{k})}$$

Define approximation error as

$$\Delta_{\overline{k}} \equiv p_t - p_{\overline{k},t}$$

A finite state representation

Proposition: The variance of the price p_t is finite

Proof: We want to show that $E\left(p_t\right)^2 < \infty$. Taking variances of both sides of the expression for the equilibrium price we get

$$\begin{aligned} E\left(p_t\right)^2 &= (\delta\gamma\lambda)^2 \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \lambda^{(i+j)} cov\left[\theta_{t+i|\cdots|t}, \theta_{t+j|\cdots|t}\right] \\ & +2\delta\gamma\lambda \sum_{j=0}^{\infty} \lambda^{j} cov\left[\theta_{t+i|\cdots|t}, \epsilon_t\right] \\ & +\left(\delta\gamma\lambda\right)^2 \sigma_\epsilon^2 \end{aligned}$$

We know from above that the covariances on right hand side are bounded and $0 < \lambda < 1$ we know that right hand side converges.

A finite state representation

Substitute in solution as function of current state state

$$p_t = \mathbf{a}\theta_{t|t}^{(0:\infty)} - \delta\gamma\lambda\epsilon_t$$

and take variances

$$\begin{aligned} E\left[p_t\right]^2 &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_i a_j cov\left[\theta_t^{(i)}, \theta_t^{(j)}\right] \\ &\quad +2\delta\gamma\lambda\sum_{j=0}^{\infty} a_j cov\left[\theta_t^{(j)}, \epsilon_t\right] \\ &\quad +\left(\delta\gamma\lambda\right)^2\sigma_\epsilon^2 \end{aligned}$$

Again, since we know that the price have finite variance, infinite sums on on right hand side must converge.

A finite state representation

The variance of the approximation error

$$\begin{aligned} E\left(\Delta_{\overline{k}}\right)^2 &= \sum_{j=\overline{k}+1}^{\infty}\sum_{i=\overline{k}+1}^{\infty} a_i a_j cov\left[\theta_t^{(i)}, \theta_t^{(j)}\right] \\ &\quad +2\delta\gamma\lambda \sum_{j=\overline{k}+1}^{\infty} a_j cov\left[\theta_t^{(j)}, \epsilon_t\right] \end{aligned}$$

must then tend to zero for the expression on previous slide to converge.

The law of motion of the expectations hierarchy

The law of motion of the expectations hierarchy

We want the form

$$\theta_{t|t}^{(0:\infty)} = M\theta_{t-1|t-1}^{(0:\infty)} + N \begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix}$$

Process for actual state

$$\theta_t = \rho\theta_{t-1} + v_t$$

Trader j's hierarchy updating equation

$$\theta_{t|t}^{(1:\infty)}(j) = M\theta_{t-1|t-1}^{(1:\infty)}(j) + K\left(S_t(j) - LM\theta_{t-1|t-1}^{(1:\infty)}(j) - Qc_t\right)$$

The law of motion of the expectations hierarchy

Signal vector is a function of the state

$$\int S_t(j) \;\; dj \;\; = \;\; L\theta_{t|t}^{(0:\infty)} + Qc_t$$

$$L \;\; = \;\; \left[\begin{array}{c} e_1' \\ \mathbf{a} \end{array} \right], \qquad Q = \left[\begin{array}{c} 0 \\ \frac{\lambda\psi}{1-\lambda\psi} \end{array} \right]$$

We can write the average hierarchy updating equation as

$$\theta_{t|t}^{(1:\infty)} \;\; = \;\; (I - KL)\, M\theta_{t-1|t-1}^{(1:\infty)} + KLM\theta_{t-1|t-1}^{(0:\infty)}$$

$$+KLN \left[\begin{array}{c} v_t \\ \epsilon_t \end{array} \right] + K \left[\begin{array}{c} 0 \\ -\delta\gamma\lambda\epsilon_t \end{array} \right]$$

The law of motion of the expectations hierarchy

Actual process and average hierarchy updating equation imply

$$M = \begin{bmatrix} \rho & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ KLM_{-} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & (I - KL)\, M_{-} \end{bmatrix}$$

$$N = \begin{bmatrix} e_1' \\ KLN_{-} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbf{0} & -K_2 \delta \gamma \lambda_{-} \end{bmatrix}$$

To solve the model, find a fixed point for M, N, a and δ

The solved model

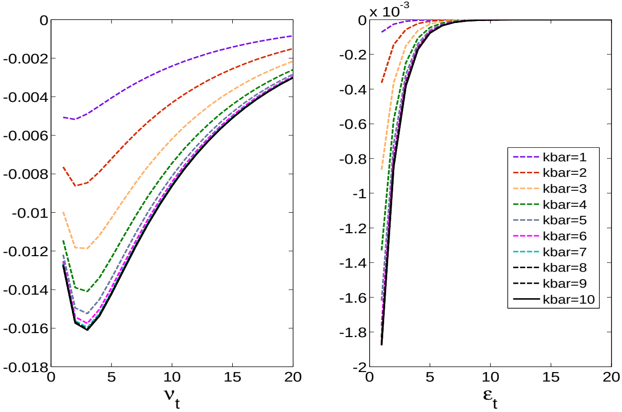
The solved model is in the form

$$\theta_{t|t}^{(0:\overline{k})} = M\theta_{t-1|t-1}^{(0:\overline{k})} + N \begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix}$$

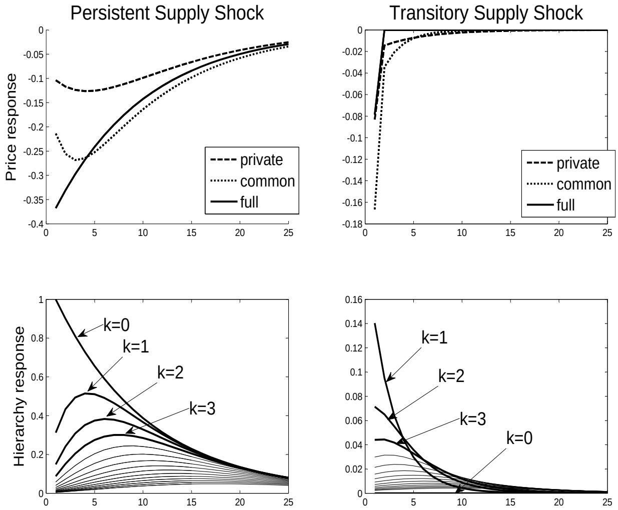
$$p_{\overline{k},t} = -\delta\gamma\lambda\epsilon_t - \delta\gamma\lambda e_1'(I - \lambda MH)^{-1}\theta_t^{(0:\overline{k})}$$

But we still need to choose $\overline{k}$

Choosing $\bar{k}$: Convergence in practice



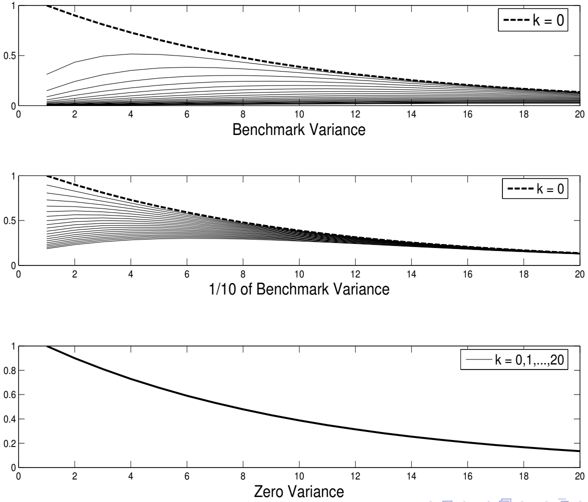
Price and hierarchy dynamics



The information revealed by prices

- Private information is not preserved in Townsend's (1983) model when equilibrium prices are observed: Sargent (1991), Kasa (2000) Pearlman and Sargent (2005)
- Walker (2007) makes similar claim about Singleton's model. But:
  - Does not prove this in Singleton's original set up
  - Makes additional assumption that the supply shock ϵ_t is directly observable
  - The price then reveals θ_t perfectly

IRF of hierarchy of θ_t and the variance of ϵ_t



The method in 3 steps

1. Impose structure on higher order expectations through common knowledge of rational expectations
2. Variance of expectations non-increasing with order of expectation
3. Impact of expectations decreasing with order of expectation

Applications

Macro

- Nimark (JME 2008, AER 2015), Graham and Wright (JEDC 2010), Melosi (2014)

Finance

- Nimark (2012), Barillas and Nimark (2014))