PRIVATE INFORMATION AND ASSET PRICING

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Private Information and Asset Pricing

These notes outline the workhorse CARA utility model that has been used to study asset pricing and portfolio choices under imperfect information. The notes present a simplified version of the set-ups in Hellwig (JET 1980) and Admati (ECTRA 1985).

1. Model Set Up

There is a continuum of competitive traders indexed by $j \in (0,1)$ who divide their wealth between a risky asset with price p and pay-off θ and a risk free asset with return that can be normalized to zero. The pay off θ on the risky asset is normally distributed

$$\theta \sim N\left(0, \sigma_\theta^2\right) \tag{1.1}$$

Initial wealth w_0^j has to be divided between the risky asset x and the safe asset m

$$w_j^0 = px_j + m_j \tag{1.2}$$

The terminal wealth of trader j is then given by

$$w_j = \theta x_j + \underbrace{\left[w_j^0 - x_j p\right]}_{\text{wealth allocated to safe asset}} \tag{1.3}$$

Trader j chooses his holdings x_j of the risky asset in order to maximize

$$E\left[-e^{-\gamma w(j)} \mid \Omega^j\right] \tag{1.4}$$

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and Ω^j is the information set of trader j. This specification is known as Constant Absolute Risk Aversion preferences, or simply CARA utility (also known as "negative exponential" utility.) As we will see, this set up delivers tractable expressions, but also have some peculiar features.

1.1. **Optimal portfolios.** If traders' signals about θ are normally distributed, expected wealth will also be conditionally normal. We can then use the following trick to solve for optimal portfolios. For any variable

$$z \sim \; N(\mu_z, \sigma_z^2) \tag{1.5}$$

the expectation of the exponential e^z is given by

$$E\left[e^z\right] = e^{(\mu_z - \frac{1}{2}\sigma_z^2)} \tag{1.6}$$

We can use this fact to evaluate the conditional expected utility

$$\begin{aligned} E\left[-e^{-\gamma w(j)} \mid \Omega^j\right] &= E\left[-e^{-\gamma\left(\theta x_j - \left[w_j^0 - x_j p\right]\right)} \mid \Omega^j\right] & (1.7) \\ &= -\exp\left(-\gamma\left(x_j E\left[\theta \mid \Omega^j\right] - \left[x_j p - w_j^0\right] - \frac{\gamma}{2} Var\left(w_j \mid \Omega^j\right)\right)\right) & (1.8) \\ &= -\exp\left(-\gamma\left(x_j E\left[\theta \mid \Omega^j\right] - \left[x_j p - w_j^0\right] - \frac{\gamma}{2} x_j^2 \sigma^2\right)\right) & (1.9) \end{aligned}$$

where σ^2 is the conditional variance of the pay off of the risky asset, i.e.

$$\sigma^2 \equiv E\left(\theta - E\left[\theta \mid \Omega^j\right]\right)^2 \tag{1.10}$$

To find the optimal portfolio, differentiate expected utility w.r.t. x_j

$$\frac{\partial E\left[-e^{-\gamma w(j)} \mid \Omega^j\right]}{\partial x_j} = -\gamma\left[E\left[\theta \mid \Omega^j\right] - p\right] + \gamma^2 x_j \sigma^2 \tag{1.11}$$

and set equal to zero and solve for trader j's demand

$$x_j = \frac{\left(E\left[\theta \mid \Omega^j\right] - p\right)}{\gamma\sigma^2} \tag{1.12}$$

This demand schedule is nice and tractable. It also has some intuitive features:

- Demand depends positively on expected return
- Demand depends negatively on conditional variance as long as risk aversion $\gamma \neq 0$
- Demand decreases in risk aversion as long as there is some uncertainty
- Trader j demand depends on his expectations about the payoff θ. If there is heterogeneity in pay off expectations, then there will be heterogeneity in portfolios.

But the demand curve for trader j in the CARA model is also peculiar in that trader j's demand does not depend on his initial wealth. This is a consequence of the constant absolute risk aversion assumption, implying that wealthy and poor investors have the same attitude to a risky bet of the same absolute magnitude. A rich and a poor investor will then hold the same amount of the risky asset as long as there are no short sales constraints, i.e. as long as we allow the poor investor to, if necessary, short the safe asset to finance his purchases of the risky asset.

1.2. **Equilibrium price.** Supply, denoted x, of the risky asset is stochastic and normally distributed

$$x \sim N\left(0, \sigma_x^2\right) \tag{1.13}$$

Equating net demand and supply

$$\int x_j = x \tag{1.14}$$

yields

$$\frac{\left(\int E\left[\theta \mid \Omega^j\right] dj - p\right)}{\gamma\sigma^2} = x \tag{1.15}$$

the equilibrium price

$$p = \left(\int E\left[\theta_t \mid \Omega_t(j)\right] \quad dj\right) - \gamma\sigma^2 x \tag{1.16}$$

1.3. **Traders' Information Sets.** The information set of trader j is given by

$$\Omega^j \equiv \{z_j, p\}$$

where

$$z_j = \theta + \eta_j : \eta_j \sim N\left(0, \sigma_\eta^2\right) \ \forall \ j \tag{1.17}$$

so that all traders are ex ante symmetric.

1.4. **Solving the model.** The model can be solved analytically. Following the steps in Admati (1985) yields a solution of the form

$$p = a\theta + bx \tag{1.18}$$

where

$$a = \left(-\gamma^{-1}\sigma_\theta^{-2} - \gamma^{-1}q^2\sigma_x^{-2} + q\right)^{-1}\left(q - \gamma^{-1}q^2\sigma_x^{-2}\right) \tag{1.19}$$

$$b = \left(-\gamma^{-1}\sigma_\theta^{-2} - \gamma^{-1}q^2\sigma_x^{-2} + q\right)^{-1}\left(1 - \gamma^{-1}q\sigma_x^{-2}\right) \tag{1.20}$$

where

$$q = -\gamma^{-1}\sigma_\eta^{-2} \tag{1.21}$$

For all parameter values, a is positive and b is negative. (Why?) Actually deriving the solution is quite involved but the principle is to conjecture that trader j's conditional expectation about the asset's pay off is a linear function of the private signal z_j and the price p

$$E\left[\theta \mid \Omega^j\right] = \begin{bmatrix} g_z & g_p \end{bmatrix} \begin{bmatrix} z_j \\ p \end{bmatrix} \tag{1.22}$$

The condition (1.15) and the conjectured form of the conditional expectation (1.22) can then be used to first find a and b in (1.18). For given a and b, g_z and g_p are given by

$$\begin{bmatrix} g_z & g_p \end{bmatrix} = \begin{bmatrix} \sigma_\theta^2 & a\sigma_\theta^2 \end{bmatrix} \begin{bmatrix} \sigma_\theta^2 + \sigma_\eta^2 & a\sigma_\theta^2 \\ a\sigma_\theta^2 & a^2\sigma_\theta^2 + b^2\sigma_x^2 \end{bmatrix}^{-1} \tag{1.23}$$

i.e. the standard projection formula.

1.5. **Properties of the solution.** There are some features of the solution that is worth mentioning. For instance, in the absence of supply shocks, the price is a function only of θ which can then be backed out perfectly from observing the price. However, as shown by Grossman (1976), this equilibrium is not the outcome of a game where agents specify demand schedules based on their private signals.

Another peculiarity of the solution is that there is a positive probability that the price of the asset is negative. This is simply a consequence of the fact that the price is a linear function of a random variable with a Gaussian distribution (which has infinite support).

1.6. **The implicit market micro structure.** In the set up above, the risky asset is traded in what can be described as a *closed book* market where agents submit *limit* and *stop* orders to a market maker that sets a *uniform* price that equates supply and demand.

- In a *closed book* market, traders cannot observe the limit and stop orders submitted by other traders, but in a *open book* all orders are observed by all market participants.
- A *limit order* specifies a quantity to be bought (or sold) if the price of the asset falls below (or increases above) a limit price. A *stop order* specifies a quantity to be sold (or bought) if the price falls below (or increases above) a limit price. Another type of order is a so called *market order*, which specifies quantity to be bought or sold at the prevailing market price.
- *Uniform* pricing means that all traders pay the same price for the asset, i.e the market maker do not price discriminate by charging a higher price to those traders who are

more optimistic about the payoff. (A trader's expectations about θ could be backed out by the market maker from observing the trader's demand schedule.)

Of these implicit assumptions, the most important is that the market is closed book: In an open book market, the average expectation could be backed out by all the traders and in extension, the true value of θ.

1.7. **Market depth.** Market depth is a concept that measures how much the price of an asset changes in response to a given change in supply. In the CARA model above, the market depth parameter is thus given by $-\gamma\sigma^2$. If risk aversion or the conditional variance σ^2 is high, an increase in supply will have a big (negative) impact on prices. This is because traders want to be compensated more when increasing their holdings of a more risky asset.

2. Multiple risky assets

The framework in Admati (1985) allows for multiple risky assets. For a vector of risky assets $\mathbf{x}$ with payoff $\mathbf{f}$ and price $\mathbf{p}$,the optimal demand for trader j with a CARA utility function is given by

$$\mathbf{x}_j = \gamma^{-1}\Sigma^{-1}\left[E\left(\mathbf{f}\mid\Omega^j\right) - \mathbf{p}\right]$$

and the equilibrium price vector is then given by

$$\mathbf{p} = \left(\int E\left[\mathbf{f} \mid \Omega^j\right] \quad dj\right) - \gamma\Sigma\mathbf{x}$$

where Σ is the conditional variance of $\mathbf{f}$. Admati (1985) provides closed form solutions for the equilibrium price, but they are not very instructive. The main economic implication is that conditional covariances now also matter for asset demand. In the limit, as the conditional variances of the assets pay offs become perfectly correlated, asset demand is no longer well-defined. Mechanically, this is because Σ is then no longer invertible. Economically, a portfolio of perfectly correlated multiple risky assets is a perfect substitute to the safe asset. With a perfect positive correlation, risk can be hedged perfectly by taking a short position in one asset and an off-setting long position in the other. With perfect negative correlation, going long in both assets also implies that risk is hedged perfectly. Traders are then indifferent between holding the safe asset and the risky assets.