DYNAMIC HIGHER ORDER EXPECTATIONS

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Abstract. In models where privately informed agents interact, agents may need to form higher order expectations, i.e. expectations of other agents' expectations. This paper develops a tractable framework for solving and analyzing linear dynamic rational expectations models in which privately informed agents form higher order expectations. The framework is used to demonstrate that the well-known problem of the infinite regress of expectations identified by Townsend (1983) can be approximated to an arbitrary accuracy with a finite dimensional representation under quite general conditions. The paper is constructive and presents a fixed point algorithm for finding an accurate solution and provides weak conditions that ensure that a fixed point exists. To help intuition, Singleton's (1987) asset pricing model with disparately informed traders is used as a vehicle for the paper.
Keywords: Dynamic Higher Order Expectations, Private Information, Asset Pricing

1. Introduction

Many economic decisions involve predicting the actions of other agents. For instance, firms in oligopolistic markets may need to predict how much production capacity their competitors will invest in and traders in financial markets may need to predict how much other traders will be willing to pay for an asset at the next trading opportunity. In settings where all agents are identical and share the same information, this becomes a trivial problem: An individual agent can predict the behavior of other agents by introspection, since all agents will choose the same action in equilibrium. The problem becomes more interesting if the common information assumption is relaxed because predicting the actions of others is then distinct from predicting ones own's actions. But since other agents face a symmetric problem, in order to predict the behavior of agents that form expectations about the actions of others, an individual agent needs to predict other agents' expectations about the actions of others, and so on, leading to the well-known infinite regress of expectations.[1] This paper develops a tractable framework for analyzing linear dynamic rational expectations models in which privately informed agents form higher order expectations. The framework is then used to

Date: First version November 2006, this version March 21, 2011. The author thanks Francisco Barillas, Vasco Carvalho, Jesus Fernandez-Villaverde, Christian Matthes, Mirko Wiederholt, Thomas Sargent and seminar participants at New York University, Birkbeck College, Goethe University Frankfurt, the 2007 annual meeting of the Society for Economic Dynamics in Prague, Australian Workshop for Macro Dynamics, Institute for International Economic Studies at Stockholm University, University of Amsterdam, and the 2009 Econometric Society meeting in San Francisco for useful comments and suggestions. Financial support from Ministerio de Ciencia e Innovacion (ECO2008-01665), Generalitat de Catalunya (2009SGR1157), Barcelona GSE Research Network and the Government of Catalonia is gratefully acknowledged.
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[1]See for instance Townsend (1983) and Sargent (1991).

demonstrate that the infinite regress of expectations can be approximated to an arbitrary accuracy with a finite dimensional representation under quite general conditions.

Conceptually, there are two distinct steps involved in deriving this result. The first is to put structure on higher order expectations by assuming that it is common knowledge that agents form model consistent, or rational, expectations. That is, all agents know that all agents know, and so on, that all agents form model consistent expectations given their information sets which gives enough structure to allow any order of expectation to be determined recursively.[2] The intuition is the following: Rationality of individual agents ensures that first order expectations are model consistent in exactly the same way that expectations are model consistent in a standard common information rational expectations model. Since this is common knowledge, the joint distribution of first order expectations and the true state is also known to all agents. Individual agents then form model consistent second order expectations by exploiting this knowledge. This argument can be applied recursively to find any order of expectation, as in the static decision settings of Morris and Shin (2002) and Woodford (2002). Here, we show how common knowledge of model consistent expectations can also be used to determine *dynamic* higher order expectations. That is, expectations today of what other agents will expect tomorrow about an event the day after tomorrow, and so on. This type of dynamic higher order expectations arise naturally in settings where privately informed agents optimize intertemporally.

Deriving the dynamics of higher order expectations does not by itself solve the problem of the infinite regress of expectations. However, common knowledge of rational expectations gives enough structure to the problem to allow us to prove the following two results: (i) The impact of expectations on the endogenous variables tends to zero as the order of expectations increases, and (ii) the variance of the approximation error introduced by only considering a finite number of orders of expectations converges to zero as the maximum order of expectations considered increases.[3] These are the main results of the paper and they can be shown to hold under quite general conditions. In the context of Singleton's (1987) asset pricing model it is demonstrated that an accurate finite dimensional representation exists under the same conditions that guarantee that a solution exists when agents are perfectly informed.

Finite numbers can still be very large, and one may ask if these results are relevant in practice. First, the paper is constructive and provides a proof that an accurate finite dimensional representation exists as well as derive an algorithm for finding it. Secondly, and again in the context of Singleton's (1987) asset pricing model, it is demonstrated numerically that the equilibrium dynamics can be captured by a low number of orders of expectation, i.e. by a vector of dimension in the single digits. This latter result may be reassuring to those who on grounds of human cognitive constraints doubt that economic agents form an infinite hierarchy of higher order expectations.

[2]In the terminology established by Harsanyi (1967-8), there is a common prior about the true state of nature and the joint probability distribution of the true state of nature and the "types". Different "types" are distinguishable only by the realizations of the private signals that they have observed in the past. The common prior then endows agents with sufficient knowledge to form model consistent expectations of the signals observed by other agents.

[3]A result with similar implications for games with countable number of players and a compact action space can be found in Weinstein and Yildiz (2007).

Introducing information imperfections into macro economics and finance is not a new idea and well-known early references include Phelps (1970), Lucas (1972, 1973, 1975), Townsend (1983), Singleton (1987) and Sargent (1991). However, recently, there has been a renewed interest in the topic and several interesting results have emerged. First, private information about quantities of common interest to all agents have been shown to introduce inertia and sluggishness of endogenous variables in settings with strategic complementarities, e.g. Woodford (2002), Morris and Shin (2006), Nimark (2008), Mackowiak and Wiederholt (2009) and Angeletos and La'o (2009). Secondly, private information may also have normative policy implications as shown by Angeletos and Pavan (2007), Lorenzoni (2009) and Paciello and Wiederholt (2011). Third, in financial markets, private information may introduce speculative behavior akin to the "beauty contest" metaphor of Keynes (1936), e.g. Allen, Morris and Shin (2006), Bacchetta and van Wincoop (2006), Kasa, Walker and Whiteman (2006), Grisse (2009), Cespa and Vives (2007).

In spite of the renewed interest, no general solution methodology with known properties has emerged for solving this class of models. This paper aims to help fill this gap and in order to understand its contribution it is useful to put it into the context of alternative solution methods used by previous literature. As a consequence of the infinite regress expectations one could characterize most existing models of private information and strategic interaction as efforts to avoid modeling higher order expectations explicitly, and instead find alternative representations where higher order expectations do not occur as state variables.[4] The most common strategy for finding a finite dimensional representation in dynamic decision models is to make private information short lived. One way to achieve this is to assume that agents pool their information between periods as in Lucas (1975) or to analyze finite horizon models as in Allen, Morris and Shin (2006) and Cespa and Vives (2007). Another way to make private information short lived to assume that all shocks are observed perfectly by all agents with a lag. This assumption was first introduced by Townsend (1983) as a way to restrict the dimension of the relevant state for 'forecasting the forecasts of others'. Optimal forecast of any variable of interest can then be constructed using projections onto the perfectly revealed state and a finite dimensional vector of signals.

This paper demonstrates how higher order expectations can be modeled explicitly in a dynamic setting without making additional assumptions to ensure that private information is short lived. The approach has at least two advantages. First, the explicit modeling of higher order expectations helps intuition as it makes the link between private information and the dynamics of endogenous variables more transparent. Secondly, since relatively few modeling compromises are needed, the solution method is more suitable than some of the alternatives for empirical work. As demonstrated by Nimark (2010) and Melosi (2011) the algorithm presented here is both flexible enough and computationally fast enough to use for likelihood based estimation of dynamic models with private information. It thus makes it feasible to empirically validate and to quantify the importance of the results from the theoretical literature mentioned above.

[4]Notable exceptions are Woodford (2002), Morris and Shin (2002) and Adam (2007) who by restricting their attention to models of static decisions are able to analyze higher order expectations explicitly.

The framework presented here can also help us understand the properties of alternative approximation approaches. Hellwig (2002) and Hellwig and Venkateswaran (2009) modifies Townsend's solution method by rewriting the equilibrium dynamics partly as an MA process and setting the lag T with which the state is revealed to be a very large number. Intuitively, it seems plausible to conjecture that in a stationary environment, the equilibrium dynamics found using this method should converge to some limit as T tends to infinity. Here we show formally that there does indeed exists a finite dimensional representation of the form used by Hellwig and Venkateswaran (2009), that as the lag T tend to infinity, converges to the true infinite dimensional solution.

Finally, a novel approach to solve dynamic models with private information that is worth mentioning and that does not rely on restricting the dimension of the state has been proposed by Kasa, Walker and Whiteman (2006) and further developed in Rondina and Walker (2010). These papers present methods that can be used to ensure that equilibrium outcomes are not perfectly revealing of the state in models where the number of signals is the same as the stochastic dimension of the model. In this class of models, Rondina and Walker (2010) show that endogenous variables can display waves of optimism and pessimism. The approach is analytically elegant and complementary to the methods proposed here, which are suitable for settings where agents face a standard filtering problem with more shocks than observables so that non-invertibility of the equilibrium process is guaranteed.

The next section defines the relevant mathematical space for analyzing dynamic higher order expectations and sets notation. This is followed by a brief presentation of the model of Singleton (1987) that will serve as a vehicle for the argument of the paper. Section 4 derives properties of higher order expectations that must hold in any equilibrium. Section 5 introduces an average expectations operator and shows how it can be used to compute equilibrium outcomes. Section 6 contains the main results of the paper. It is here that the approximation results are presented, demonstrating that a finite number of orders of expectations are sufficient for an arbitrarily accurate representation of equilibrium. Section 7 presents an algorithm to find the equilibrium and proves that an equilibrium exists under quite general conditions. Section 8 presents properties of the solved model and shows that in practise, only a low number of orders of expectations are necessary as equilibrium dynamics converge rapidly as the maximum order of expectation is increased. Section 9 demonstrates that the equilibrium dynamics can be recast in the form used by Hellwig and Venkateswaran (2009) and Section 10 concludes. The Appendix contains some proofs left out of the main text.

2. Preliminaries

Before analyzing the dynamics of higher order expectations, it is necessary to invest a little in notational machinery as well as to define exactly what is meant by a higher order expectation.

2.1. **The inner product space L^2.** In the model presented in the next section, the signals that traders observe and their expectations of fundamentals and endogenous variables are elements of the inner product space L^2, which we now define.

Definition 1. *(The inner-product space L^2.) The inner product space L^2 is the collection of all random variables X with finite variance*

$$EX^2 < \infty \tag{2.1}$$

and with inner-product

$$\langle X, Y \rangle \equiv E(XY) : X, Y \in L^2 \tag{2.2}$$

Definition 2. *Let Ω be a subspace of L^2. An orthogonal projection of X onto Ω , denoted $\mathcal{P}_\Omega X$, is the unique element in L^2 satisfying*

$$\langle X - \mathcal{P}_\Omega X, \omega \rangle = 0 \tag{2.3}$$

for any $\omega \in \Omega$.

In a linear model with Gaussian shocks, conditional expectations are equivalent to orthogonal projections. The equality

$$E(X \mid \Omega) = \mathcal{P}_\Omega X \tag{2.4}$$

thus implies that the conditional expectations in the model share the properties of orthogonal projections in L^2. (For more details, see Brockwell and Davis 2006.)

2.2. **Defining higher order expectations.** There is a continuum of agents indexed by $j \in (0, 1)$. Agent j's first order expectation of a variable $\theta_t \in L^2$ conditional on his period t information set $\Omega_t(j)$ is denoted as

$$\theta_t^{(1)}(j) \equiv E[\theta_t \mid \Omega_t(j)] \tag{2.5}$$

The average first order expectation $\theta_t^{(1)}$ is obtained by taking averages of (2.5) across agents

$$\theta_t^{(1)} \equiv \int E[\theta_t \mid \Omega_t(j)] \quad dj \tag{2.6}$$

The average second order expectation is obtained by taking the average of agents' expectations of (2.6)

$$\theta_t^{(2)} \equiv \int E\left[\theta_t^{(1)} \mid \Omega_t(j)\right] \; dj \tag{2.7}$$

and so on so that the k^{th} order expectation of θ_t is given by

$$\theta_t^{(k)} \equiv \int E\left[\theta_t^{(k-1)} \mid \Omega_t(j)\right] \quad dj \tag{2.8}$$

It is sometimes useful to define the zero order expectation of θ_t as the actual value of the variable

$$\theta_t^{(0)} \equiv \theta_t \tag{2.9}$$

Full information rational expectations implies that the variable θ_t is common knowledge so that $\theta_t = \theta_t^{(k)} : k = 1, 2, ...$ for all periods t. We call a sequence of expectations, for instance from order zero to k, a *hierarchy* of expectations from order zero to k. Vectors consisting of a hierarchy of expectations are denoted

$$\theta_t^{(0:k)} = \begin{bmatrix} \theta_t^{(0)} & \theta_t^{(1)} & \dots & \theta_t^{(k)} \end{bmatrix}' \tag{2.10}$$

2.2.1. *Expectations about future expectations.* In later sections, it will prove useful to also have a notation for the average expectation held in period t of the average expectation held in period $t+1$ of the value of a variable in period $t+2$, and so on. For that purpose, we define the following notation. The first order expectation in period t of θ_{t+1} is defined as

$$\theta^{(1)}_{t+1|t} \equiv \int E\left[\theta_{t+1} \mid \Omega_t(j)\right] \quad dj \tag{2.11}$$

Similarly, the average expectation in period t of the average expectation in period $t+1$ of θ_{t+2} is defined as

$$\theta^{(2)}_{t+2|t+1|t} \equiv \int E\left[\theta^{(1)}_{t+2|t+1} \mid \Omega_t(j)\right] \quad dj \tag{2.12}$$

Generalizing this notation

$$\theta^{(k)}_{t+k|\cdots|t} \equiv \int E\left[\theta^{(k-1)}_{t+k|\cdots|t+1} \mid \Omega_t(j)\right] \ dj \tag{2.13}$$

3. The Singleton Asset Pricing Model

This section presents a version of the model of Singleton (1987) with disparately informed traders that will serve as the vehicle for the argument in the rest of the paper. Singleton presents and solves a number of models that differ slightly in their patterns of persistence and assumed structural parameter values. In what he refers to as Models 1-7, the unobservable fundamental process follows an MA(2) process and in Models 8-12 it follows an AR(1). In this first class of models, a finite dimensional state representation can be found without making strong assumptions about the revelation of the shocks since a private signal about a MA(2) process does not carry information that is useful for forecasts beyond a two period horizon. Private information about an AR(1) process on the other hand is long lived. To solve the second class of models, Singleton assumes that the innovations to the AR(1) process are perfectly and publicly observed with a two period lag. This allows him to derive a finite dimensional state representation. The rest of this paper uses the same set up as in Singleton's Models 8-12 as a vehicle to show how dynamic models with private information can be solved *without* assuming that the shocks to the hidden process ever become common knowledge.

3.1. **Model Set Up.** There is a continuum of competitive traders indexed by $j \in (0,1)$ who at time t divide their wealth between a risky asset with price p_t and coupon payment c_t and a risk free asset with return $\overline{r}$. The wealth of trader j then evolves according to

$$w_{t+1}(j) = z_t(j)\left[p_{t+1} + c_{t+1}\right] - \left[z_t(j)p_t - w_t(j)\right](1+\overline{r}) \tag{3.1}$$

where $z_t(j)$ is the asset holdings of trader j who chooses his portfolio to maximize

$$E\left[-e^{-\gamma w_{t+1}(j)} \mid \Omega_t(j)\right] \tag{3.2}$$

and $\Omega_t(j)$ is the information set of trader j at time t (defined below). The coupon payments follow the known autoregressive process

$$c_t = \overline{c} + \psi c_{t-1} + u_t : u_t \sim N\left(0, \sigma^2_u\right) \tag{3.3}$$

Maximizing (3.2) subject to (3.1) yields agent j's optimal demand for the risky asset

$$z_t^d(j) = \frac{\left(E\left[p_{t+1} \mid \Omega_t(j)\right] - (1+\overline{r})\, p_t\right) + (\overline{c} + \psi c_t)}{\gamma\delta} \tag{3.4}$$

where δ is the conditional variance of $(p_{t+1} + c_{t+1})$. The supply of the asset at time t, z_t^s, depends linearly on the price p_t and additively on the persistent stochastic shock θ_t and the i.i.d. disturbance ϵ_t

$$z_t^s = \xi p_t + \theta_t + \epsilon_t : \epsilon_t \sim N\left(0, \sigma_\epsilon^2\right) \tag{3.5}$$

$$\theta_t = \rho\theta_{t-1} + v_t : v_t \sim N\left(0, \sigma_v^2\right) \tag{3.6}$$

Equating net demand and supply

$$\int z_t^d(j) = z_t^s \tag{3.7}$$

yields the equilibrium price

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid \Omega_t(j)\right] \quad dj \right) + \lambda\psi c_t - \delta\gamma\lambda \left[\theta_t + \epsilon_t\right] \tag{3.8}$$

where

$$\lambda \equiv \frac{1}{\xi\gamma\delta + (1+\overline{r})}. \tag{3.9}$$

For later reference, note that $0 < \lambda < (1+\overline{r})^{-1}$.

3.2. **Traders' Information Sets.** The basic structure of the model described above is identical to Model 8-12 in Singleton (1987). Where this paper differ from Singleton's is in the assumption on what traders can observe. In Singleton's paper the information set $\Omega_t^S(j)$ of trader j at time t is given by

$$\Omega_t^S(j) = \{s_{t-T}(j), p_{t-T}, c_{t-T} : T \geq 0; v_{t-T}, \epsilon_{t-T} : T \geq 2\} \tag{3.10}$$

where

$$s_t(j) = \theta_t + \eta_t(j) : \eta_t(j) \sim N\left(0, \sigma_\eta^2\right) \ \forall \ \ j \tag{3.11}$$

Each trader observes the price of the asset, p_t, and the coupon payment, c_t, perfectly. The persistent component θ_t of the supply process is not perfectly revealed by the observation of the price due to the unobservable transitory supply shock ϵ_t. The transitory supply shock ϵ_t thus serves the same purpose here as the noise traders do in Admati (1985). Trader j also observes a private signal $s_t(j)$ of the persistent supply process θ_t and it is due to the private measurement error $\eta_t(j)$ that the need to 'forecast the forecasts of others' arises. Singleton uses a similar method to overcome the infinite dimension of the state as Townsend (1983), i.e. he assumes that the shocks to the supply process become known to all traders after a finite number of periods (which in Singleton's case is after two periods). This allows for a finite dimensional time series representation of the model.

While the assumption of public revelation of shocks with a lag is convenient from a modeling perspective, it is not an assumption that is always realistic. We want to solve the model without imposing that all shocks are observed perfectly after a finite number of periods. The information set of our trader is therefore given by

$$\Omega_t(j) = \{s_{t-T}(j), p_{t-T}, c_{t-T} : T \geq 0\} \tag{3.12}$$

Traders thus form expectations about the future price of the asset by observing the private signal $s_t(j)$, the commonly observable price p_t and the coupon payment c_t. It is common knowledge that all traders choose their portfolio to maximize (3.2) subject to the structural equations (3.3) - (3.6).

3.3. **The full information solution.** To solve the model we need to integrate out the average expectations term $\int E\left[p_{t+1} \mid \Omega_t(j)\right] \ dj$ from equation (3.8). Under full information, this could be done by iterating (3.8) forward

$$p_t = \sum_{k=1}^{\infty} \lambda^k E\left(c_{t+k} \mid c_t\right) - \delta\gamma\lambda \sum_{k=0}^{\infty} \lambda^k E\left(\theta_{t+k} \mid \theta_t\right) - \delta\gamma\lambda\epsilon_t. \tag{3.13}$$

Using the law of iterated expectations, (3.13) then simplifies to

$$p_t = \frac{\lambda\psi}{1-\lambda\psi} c_t - \frac{\delta\gamma\lambda}{1-\lambda\rho}\theta_t - \delta\gamma\lambda\epsilon_t \tag{3.14}$$

if $|\lambda\psi| < 1$ and $|\lambda\rho| < 1$.

3.4. **A complication.** With privately informed traders, we can still use forward substitution of the Euler equation (3.8). This yields the equilibrium price as a function of higher order expectations of future values of the persistent supply process θ_t

$$p_t = \frac{\lambda\psi}{1-\lambda\psi} c_t - \delta\gamma\lambda \sum_{k=0}^{\infty} \lambda^k \theta^{(k)}_{t+k|\cdots|t} - \delta\gamma\lambda\epsilon_t \tag{3.15}$$

where we used the notation for higher order expectations of future values of θ_t defined in Section 2.2.1. The current price of the asset thus depends on the average expectation in period t of θ_{t+1}, the average expectation in period t of the average expectation in period $t+1$ of θ_{t+2} and so on. As has been noted before, e.g. Allen, Morris and Shin (2006), average higher order expectations, i.e. expectations about other agent's expectations generally differ from average first order expectations and we cannot use the law of iterated expectations to integrate out the higher order expectations in the price equation (3.15). To see why, note that the law of iterated expectations can loosely speaking be attributed to the fact that agents do not believe that they have 'incorrect' expectations so that they do not expect to revise their own expectations in a particular direction. That is, first order expectations are martingales. The same is not true about expectations about other agents' expectations. For instance, an investor may believe that the average 'market expectation' of the fundamental value of an asset is incorrect, but as more information becomes available to others over time the 'market expectation' will be revised towards what the investor believes is the asset's true value. It is the fact that it can be rational to expect others to revise their expectations in a certain direction that makes the law of iterated expectations inapplicable to higher order expectations. It is also this fact that makes the dynamics of models with private information interesting.

3.5. **The strategy.** The rest of the paper is devoted to finding a finite dimensional representation of the equilibrium price (3.15) of the form

$$p_{\overline{k},t} = \frac{\lambda\psi}{1-\lambda\psi}c_t - \mathbf{a}_{\overline{k}}\theta_t^{(0:\overline{k})} - \delta\gamma\lambda\epsilon_t \tag{3.16}$$

that is arbitrarily close to the solution to the equilibrium price (3.15) and where $\mathbf{a}'_{\overline{k}}$ and $\theta_t^{(0:\overline{k})}$ are finite dimensional vectors. We will demonstrate that the discounted sum of higher order expectations of all future values of θ_t in (3.15) can be approximated by a linear function of a finite number of orders of expectations of the current value of θ_t so that the variance of approximation error $\Delta_{\overline{k},t}$ in

$$\delta\gamma\lambda\sum_{k=0}^{\infty}\lambda^k\theta_{t+k|\cdots|t}^{(k)} \equiv \mathbf{a}_{\overline{k}}\theta_t^{(0:\overline{k})} + \Delta_{\overline{k},t} \tag{3.17}$$

can be made arbitrarily small by choosing a large enough $\overline{k}$. To do so, we will conjecture (and later verify) that there exists a law of motion for the hierarchy of higher order expectations of the current value of θ_t of the form

$$\theta_t^{(0:\overline{k})} = M\theta_{t-1}^{(0:\overline{k})} + Nw_t : w_t \sim N(0,I) \tag{3.18}$$

The solution will then consist of the equilibrium price (3.16) and the law of motion for the state (3.18).

The plan from here on is the following. First we will derive some properties of higher order expectations that must hold in any equilibrium. We then show how the price of the asset can be expressed as a function of the conjectured law of motion (3.18). This will give enough structure to the problem to show that there exists a representation with a finite number of orders of expectations that can be made arbitrarily close to the infinite dimensional representation. These results are quite general in that they will hold under the same conditions that guarantee that a stable solution exists under full information, i.e. that $|\lambda\rho| < 1$.

4. Equilibrium properties of higher order expectations

It is possible to characterize some properties of higher order expectations using only that it is common knowledge that agents form expectations rationally. The properties derived in this section will be important for the approximation results presented in Section 6 below, but they also help develop intuition by making the link between common knowledge of rational expectations and the properties of higher order expectations explicit.

4.1. **First order expectations.** We start by establishing some properties of first order expectations. This may seem pedantic, since properties of first order expectations are well known. However, this will lay the groundwork for recursively deriving similar, but more interesting, properties of higher order expectations. We start by defining a useful subspace of L^2.

Definition 3. *The (closed) subspace* $\Omega_t(j) \equiv \overline{sp}\left\{s_{t-T}(j), p_{t-T}, c_{t-T} : T \geq 0\right\}$ *is the space spanned by the history of variables observed by trader* j *at period* t*. Projections onto* $\Omega_t(j)$ *are denoted* $\mathcal{P}_{t,j}$.

From the projection theorem (e.g. Brockwell and Davis (2006)) we then know that there exist an element $\theta_t^{(1)}(j) \in L^2$ such that

$$\left\langle \theta_t - \theta_t^{(1)}(j), \omega_j \right\rangle = 0 \;\; \forall \;\; \omega_j \in \Omega_t(j) \tag{4.1}$$

that is, there exists a minimum variance expectation of θ_t conditional on trader j's information set. Given the linear structure of the model, past realizations of v_t, ϵ_t and $\eta_t(j)$ form an orthogonal basis for the subspace $\Omega_t(j)$. The conditional expectation $E\left[\theta_t \mid \Omega_t(j)\right]$ thus has a representation of the form

$$\theta_t^{(1)}(j) = A(L)v_t + B(L)\epsilon_t + C(L)\eta_t(j) \tag{4.2}$$

where by the ex ante symmetry of traders, the (potentially infinite order) lag polynomials $A(L)$, $B(L)$ and $C(L)$ are common across traders. Expectations will differ across traders only because of different realizations of the idiosyncratic noise shocks $\eta_t(j)$.

4.2. **The variance of first order expectations.** Here, the orthogonality property (4.1) and the representation (4.2) will be used to prove that the variance of average higher order expectations are bounded by the variance of lower order expectations. This result will later be used for the approximation results in Section 6 as well as for the existence results in Section 7. We start by showing that the variance of trader j's first order expectations of θ_t is bounded by the variance of the actual process θ_t.

Lemma 1. *The variance of trader* j*'s expectation of* θ_t *is bounded by the variance of* θ_t*, i.e.*

$$E\left[\theta_t\right]^2 \geq E\left[\theta_t^{(1)}(j)\right]^2 \tag{4.3}$$

Proof. Define trader j's first order expectation error $\varepsilon_t^{(1)}(j)$ as

$$\theta_t - \theta_t^{(1)}(j) \equiv \varepsilon_t^{(1)}(j) \tag{4.4}$$

and rearrange

$$\theta_t \equiv \theta_t^{(1)}(j) + \varepsilon_t^{(1)}(j) \tag{4.5}$$

The variance of the l.h.s. is $E\left[\theta_t\right]^2$. By (4.1), the error $\varepsilon_t^{(1)}(j)$ is orthogonal to $\theta_t^{(1)}(j) \in \Omega_t(j)$ so the variance of the r.h.s. is simply the sum of the variances of the individual terms, which gives the equality

$$E\left[\theta_t\right]^2 = E\left[\theta_t^{(1)}(j)\right]^2 + E\left[\varepsilon_t^{(1)}(j)\right]^2 \tag{4.6}$$

The proof then follows from the fact that variances are non-negative

$$0 \leq E\left[\varepsilon_t^{(1)}(j)\right]^2 \tag{4.7}$$

so that

$$E\left[\theta_t\right]^2 \geq E\left[\theta_t^{(1)}(j)\right]^2 \tag{4.8}$$

$\square$

According to the representation (4.2), trader j's expectation has both a common and idiosyncratic component. The fact that the idiosyncratic component is orthogonal to the common component allows us to prove our next result.

Lemma 2. *The variance of the average expectation of θ_t is bounded by the variance of θ_t, i.e.*

$$E\left[\theta_t\right]^2 \geq E\left[\theta_t^{(1)}\right]^2 \tag{4.9}$$

Proof. The representation (4.2) implies that the variance of trader j's first order expectations is the sum of the variances of the terms in the MA representation

$$E\left[\theta_t^{(1)}(j)\right]^2 = E\left[A(L)v_t\right]^2 + E\left[B(L)\epsilon_t\right]^2 + E\left[C(L)\eta_t(j)\right]^2 \tag{4.10}$$

Since $\int \eta_t(j)dj = 0\forall t$ the average first order expectation is simply

$$\begin{aligned} \theta_t^{(1)} &= A(L)v_t + B(L)\epsilon_t + \int C(L)\eta_t(j)dj && (4.11)\\ &= A(L)v_t + B(L)\epsilon_t && (4.12) \end{aligned}$$

with variance

$$E\left[\theta_t^{(1)}\right]^2 = E\left[A(L)v_t\right]^2 + E\left[B(L)\epsilon_t\right]^2 \tag{4.13}$$

Combining (4.8) and (4.10)

$$\begin{aligned} E\left[\theta_t\right]^2 &\geq E\left[\theta_t^{(1)}(j)\right]^2 && (4.14)\\ &= E\left[A(L)v_t\right]^2 + E\left[B(L)\epsilon_t\right]^2 + E\left[C(L)\eta_t(j)\right]^2 && (4.15)\\ &\geq E\left[A(L)v_t\right]^2 + E\left[B(L)\epsilon_t\right]^2 && (4.16)\\ &= E\left[\theta_t^{(1)}\right]^2 && (4.17) \end{aligned}$$

gives the desired result where the third line follows from the fact that $0 \leq E\left[C(L)\eta_t(j)\right]^2$ and the last equality is from (4.13) $\square$

4.3. **Variance bounds for higher order expectations.**

Proposition 1. *The variance of higher order expectations of θ_t are bounded by the variance of lower order expectations, i.e.*

$$E\left[\theta_t^{(k)}\right]^2 \geq E\left[\theta_t^{(k+1)}\right]^2 \tag{4.18}$$

Proof. To prove the proposition, replace the definition of trader j's first order expectations error $\varepsilon_t^{(1)}(j)$ in the proof of Lemma 1 with the definition of the k order expectation error

$$\theta_t^{(k-1)} - \theta_t^{(k)}(j) \equiv \varepsilon_t^{(k)}(j) \tag{4.19}$$

Noting that the k order error $\varepsilon_t^{(k)}(j)$ is orthogonal to $\theta_t^{(k)}(j) \in \Omega_t(j)$ allows for recursively establishing the proposition for $k = 2, 3, ...$by following the same steps as in the proofs of Lemma 1 and 2. $\square$

It is straightforward to extend this result to higher order expectations of future values of θ_t.

Proposition 2. *The variance of higher order expectations of future expectations of θ_t are bounded by the variance of lower order expectations, i.e.*

$$E\left[\theta^{(k-1)}_{t+k|\cdots|t+1}\right]^2 \geq E\left[\theta^{(k)}_{t+k|\cdots|t}\right]^2 \tag{4.20}$$

Proof. To prove the proposition, replace the definition of trader j's first order expectations error $\varepsilon^{(1)}_t(j)$ in the proof of Lemma 1 with the definition of the period t first order expectation error about θ_{t+1}

$$\theta_{t+1} - \theta^{(1)}_{t+1|t}(j) \equiv \varepsilon^{(1)}_{t+1|t}(j) \tag{4.21}$$

Using the same steps as in Lemma 1and Lemma 2, it can be established that

$$E\left[\theta_{t+1}\right]^2 \geq E\left[\theta^{(1)}_{t+1|t}\right]^2 \tag{4.22}$$

so that average first order expectation about future values of θ are also bounded since $E\left[\theta_{t+1}\right]^2 = E\left[\theta_t\right]^2$. Defining a k order future expectation error

$$\theta^{(k-1)}_{t+k|\cdots|t+1} - \theta^{(k)}_{t+k|\cdots|t}(j) \equiv \varepsilon^{(k)}_{t+k|\cdots|t}(j) \tag{4.23}$$

and again repeating the steps of Lemma 1 and 2 establishes the desired result for $k = 2, 3, ...$ $\square$

As noted above, these results will prove useful for the purpose of finding an accurate solution to the model, but they also illustrate well how the assumption of common knowledge of rational (i.e. model consistent) expectations lets us derive properties of higher order expectations. It is the fact that first order expectations are formed rationally and that this is common knowledge that allows us to derive the variance bounds above. In the absence of common knowledge of model consistent expectations, we would have to make alternative assumptions about how traders in the model believe that other traders form expectations in order to determine how traders form second order expectations. Whether the variance bounds derived above would hold or not, would then depend on the properties of the second order beliefs about how other traders form expectations.

4.4. **Properties of the law of motion for higher order expectations.** In the solution algorithm proposed in Section 7, we conjecture (and verify) that the hierarchy of higher order expectations of θ_t follows a vector autoregressive process of the form

$$\theta^{(0:\overline{k})}_t = M\theta^{(0:\overline{k})}_{t-1} + Nw_t : w_t \sim N(0, I) \tag{4.24}$$

We now prove that (4.24) must be a stable process.

Proposition 3. *If θ_t follows a stable process, i.e. if $|\rho| < 1$, then common knowledge of rational expectations implies that $max\left|eig\left(M\right)\right| < 1$.*

Proof. The proof is by contradiction and is a direct corollary of Proposition 1. Consider the case if $max\left|eigM\right| = 1$. This implies that at least one $k \neq 0$ order of expectation of θ_t has a unit root and as a consequence that the variance of at least one k' order of expectation of

θ_t is increasing without bound in t. But from Proposition 1 we know that the variances of higher order expectations are bounded by the variance of θ_t and therefore (4.24) must be a stable process, i.e. $max\,|eig\,(M)| < 1$. □

In this section, we have derived some properties of higher order expectations that must hold in any equilibrium when it is common knowledge that traders form expectations rationally. Specifically, we showed that the variance of higher order expectations are bounded using orthogonality properties of expectation errors. While based on a simple insight, these results will later turn out to be very useful for both deriving an accurate finite dimensional solution as well as for demonstrating that such a solution exists.

5. The equilibrium price

This section demonstrates how a simple matrix operator can be used to compute the equilibrium price for a given law of motion of the hierarchy of higher order expectations. The law of motion for the hierarchy of expectations is derived in the Section 7.

5.1. An average higher order expectations operator.

To compute the higher order expectations we will use the linear operator $H : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined so that

$$\theta_t^{(k+1:\infty)} = H\theta_t^{(k:\infty)} \tag{5.1}$$

That is, H applied to a hierarchy of expectations move the hierarchy one step up in order of expectations. If the state of the economy is given by $\theta_t^{(0:\infty)}$ then the average expectations of the true state is given by $H\theta_t^{(0:\infty)}$ and the operator H thus annihilates the first element of a vector of higher order expectations. The operator is given by the matrix

$$H \equiv \begin{bmatrix} \mathbf{0} & I_\infty \end{bmatrix} \tag{5.2}$$

where I_∞ is the identity matrix.[5]

5.2. Equilibrium asset prices.

We can now derive an explicit expression for the equilibrium price of the asset. Given the conjectured law of motion (3.18) and the higher order expectations operator we can now compute the higher order expectations of the future values of θ_t in the forward iteration (3.15) of the price Euler equation (3.8).

The one step ahead average expectation of θ_t is simply given by first applying H to the complete hierarchy of expectation to get the average expectation of the state and then apply M to form the average expectation of the state in the next period. The average expectation in period t of the value of the persistent supply shock θ_t is then given by

$$\int E\left[\theta_{t+1} \mid \Omega_t(j)\right] \quad dj = e_1' M H \theta_t^{(0:\infty)} \tag{5.3}$$

[5] Allen, Morris and Shin (2006) defines an *average belief operator* $\overline{E} : \mathbb{R}^2 \to \mathbb{R}^2$. The operator $\overline{E}$ maps the average k order expectations of the average signal vector into $k+1$ order expectations of the same vector and can be used to compute higher order expectations of the state since the static setting results in a proportional relationship between higher order beliefs. In our model, the elements of N in the law of motion (3.18) could be generated by a similar operator if θ_t was a non-persistent process.

where e_1 is a vector with 1 in the first element and zeros elsewhere. Using similar reasoning, the expectation in period t of the average expectation in period $t+1$ of θ_{t+2} is then given by

$$\theta^{(2)}_{t+2|t+1|t} = e_1' \left(MH\right)^2 \theta_t^{(0:\infty)} \tag{5.4}$$

More generally, the k order expectation of θ_{t+k} is given by

$$\theta^{(k)}_{t+k|\cdots|t} = e_1' \left(MH\right)^k \theta_t^{(0:\infty)} \tag{5.5}$$

Substituting (5.5) into (3.15) gives the equilibrium price p_t as a function of the period t hierarchy of higher order expectations of θ_t as

$$p_t = -\delta\gamma\lambda \sum_{k=0}^{\infty} e_1' \left(\lambda MH\right)^k \theta_t^{(0:\infty)} - \delta\gamma\lambda\epsilon_t + \frac{\lambda\psi}{1-\lambda\psi} c_t \tag{5.6}$$

where we used that $\theta_t = e_1'\theta_t^{(0:\infty)}$. The relationship between M and MH is thus analogous to that of physical and risk neutral dynamics in the finance literature. Of course, the interpretation is different. In the standard framework, fundamentals follows the physical measure but assets are priced as if traders are risk neutral and fundamentals follow the risk neutral measure. Here, fundamentals follow the process described by M which is thus analogous to the physical measure, but the asset is priced as if traders observed the true state and the fundamentals followed a VAR process with coefficient matrix MH.

6. A Finite Dimensional Approximation

In the previous sections, several properties of higher order expectations were derived that must hold in equilibrium. Though some of these properties may be interesting per se, here we show how they can be used to prove a practical result: The equilibrium characterized by an infinite number of orders of expectations can be approximated to an arbitrary accuracy by a finite dimensional system. That is, for practical purposes, we do not need to consider the complete hierarchy of expectation, but instead we can find a maximum (and finite) order of expectation that we need to consider, for any desired degree of accuracy. We denote this maximum order of expectation $\overline{k}$.

Two results are proved formally here. First, we show that the weight on higher order expectations tend to zero as the order of expectation increases. Secondly, we show that the variance of the approximation error tends to zero as we increase the maximum number of orders of expectations $\overline{k}$. Both results are derived using a similar technique. First, we define an infinite series indexed by the number of orders of expectations. We then show that the series converges. Since convergence of an infinite series implies that the individual elements in the sequence tend to zero (while the converse is not generally true), convergence of a series indexed by the maximum order of expectation considered implies that the cumulative effect of terms depending on orders of expectations higher than $\overline{k}$ tend to zero.

6.1. **The diminishing impact of higher order expectations.** The solved model will deliver an expression for the equilibrium price p_t as a function of the current hierarchy of expectations about θ_t of the form

$$p_{\overline{k},t} = \mathbf{a}_{\overline{k}}\theta_t^{(0:\overline{k})} - \delta\gamma\lambda\epsilon_t + \frac{\lambda\psi}{1-\lambda\psi}c_t \tag{6.1}$$

where $\mathbf{a}_{\overline{k}}$ is a row vector with elements defined as

$$\mathbf{a}_{\overline{k}} \equiv \left[\begin{array}{cccc} a_0 & a_1 & \cdots & a_{\overline{k}} \end{array}\right]$$

From the price equation (3.8) we already know that $a_0 = -\delta\gamma\lambda$. The next proposition establishes that as $\overline{k} \to \infty$ the coefficient $a_{\overline{k}}$ tends to zero.

Proposition 4. *For $|\lambda\rho| < 1$,there exists a finite number $\overline{k}$ such that*

$$|a_k + a_{k+1} + ... + a_{k+n-1} + a_{k+n}| < \varepsilon \tag{6.2}$$

for any $\varepsilon > 0$, for all $n \geq 1$ and $k > \overline{k}$.

Proof. The result is an immediate implication of the fact that $\{\Sigma a_k\}_{k=0}^{\infty}$ is a convergent series, which we now establish. To do so, we will use the fact that in the special case when $\theta_t = \theta_t^{(k)} = 1 \forall k$ the equilibrium price equals the sum of the elements in the row vector $\mathbf{a}$.

First, note that (if by chance), all orders of expectations coincide so that $\theta_t = \theta_t^{(k)}\ \forall\ k$ then common knowledge of rational expectations implies that higher order expectations about the future values of θ_t must coincide with first order expectations. That is, if there is agreement about the current state, there must also be agreement about expected future states so that

$$\begin{aligned} e_1'\left(MH\right)^k\theta_t^{(0:\infty)} &= e_1'\left(MH\right)^k\left[\begin{array}{c}\theta_t\\ \theta_t\\ \vdots\\ \theta_t\end{array}\right] && (6.3)\\ &= \rho^k\theta_t : k = 0,1,2... && (6.4)\end{aligned}$$

Now, setting $\theta_t^{(k)} = 1 : k = 0,1,2...$

$$e_1'\left(MH\right)^k \times \mathbf{1}_\infty = \rho^k : k = 0,1,2... \tag{6.5}$$

and substituting the right hand side of (6.5) into the price equation (5.6) gives

$$p_t = -\delta\gamma\lambda\sum_{k=0}^{\infty}\left(\lambda\rho\right)^k - \delta\gamma\lambda\epsilon_t + \frac{\lambda\psi}{1-\lambda\psi}c_t \tag{6.6}$$

or that

$$\begin{aligned} -\delta\gamma\lambda\sum_{k=0}^{\infty}e_1'\left(\lambda MH\right)^k \times \mathbf{1}_\infty &= -\delta\gamma\lambda\sum_{k=0}^{\infty}\left(\lambda\rho\right)^k && (6.7)\\ &= -\frac{\delta\gamma\lambda}{1-\lambda\rho} && (6.8)\end{aligned}$$

Intuitively, if all orders of expectations coincide then the price must equal the full information price (adjusted for the appropriate private information values of λ and δ). By definition, the l.h.s. of (6.7) equals the infinite sum of the elements of the row vector $\mathbf{a}_\infty$, i.e.

$$-\delta\gamma\lambda \sum_{k=0}^{\infty} e_1' \left(\lambda MH\right)^k \times \mathbf{1}_\infty \equiv \mathbf{a}_\infty \times \mathbf{1}_\infty \tag{6.9}$$

$$= \sum_{k=0}^{\infty} a_k \tag{6.10}$$

Combining (6.8) and (6.10) establishes the limit of $\{\Sigma a_k\}_{k=0}^{\infty}$ as

$$\sum_{k=0}^{\infty} a_k = -\frac{\delta\gamma\lambda}{1-\lambda\rho} \tag{6.11}$$

which is finite. Since the infinite series (6.11) converges, there exists a number $\overline{k}$ such that

$$|a_k + a_{k+1} + ... + a_{k+n-1} + a_{k+n}| < \varepsilon \text{ for all } n \geq 1 \text{ and } k > \overline{k} \tag{6.12}$$

(see for instance Ok 2007). □

Proposition 4 thus establishes that the coefficients a_k that multiply the k order expectation in the conjectured solution tend to zero as the order of expectation increases, and that this will hold under the same conditions that guarantee that a stable solution to the full information model exists, i.e. that $|\lambda\rho| < 1$.

6.2. **The variance of the approximation error.** Above, we demonstrated that the impact of expectations on the price tend to zero as the order of expectations increases. Combined with the fact that the variance of higher order expectations are bounded, one might conjecture that the variance of the contribution of the higher order expectation to the price also tend to zero as the order of expectation increases. Here, we will now demonstrate that this is indeed the case but using a more direct approach that does not involve using the result of Proposition 4 above. Instead, we will define a particular convergent series (again indexed by $\overline{k}$) so that the remainder of the sum corresponds to the variance of the approximation error. Since the series converges, the remainder can be made arbitrarily small for large enough $\overline{k}$. To prove this result, we will need the following lemma.

Lemma 3. *The variance of the price p_t is finite.*

Proof. The proof uses that the higher order expectations about future expectations of future values of θ_t in the price equation (3.15) are discounted by $|\rho| < 1$ and have finite variances. The complete proof can be found in the Appendix. □

Definition 4. *The approximation error $\Delta_{\overline{k},t}$ associated with considering only $\overline{k}$ orders of expectations is defined as*

$$\Delta_{\overline{k},t} \equiv p_t - p_{\overline{k},t} \tag{6.13}$$

where

$$p_t = \mathbf{a}_{\overline{k}} \theta_t^{(0:\infty)} - \delta\gamma\lambda\epsilon_t + \frac{\lambda\psi}{1-\lambda\psi} c_t \tag{6.14}$$

and

$$p_{\overline{k},t} = \mathbf{a}_{\overline{k}} \theta_t^{(0:\overline{k})} - \delta\gamma\lambda\epsilon_t + \frac{\lambda\psi}{1-\lambda\psi} c_t \tag{6.15}$$

so that

$$\Delta_{\overline{k},t} = \left(\mathbf{a} - \left[\begin{array}{cc} \mathbf{a}_{\overline{k}} & \mathbf{0} \end{array}\right]\right) \theta_t^{(0:\infty)} \tag{6.16}$$

Proposition 5. *The variance of the approximation error $\Delta_{\overline{k},t}$ tends to zero as $\overline{k}$ tends to infinity.*

Proof. First, define the sequence$\{z_{\overline{k}}\}$

$$\{z_{\overline{k}}\} = \left\{ \sum_{j=0}^{\overline{k}} \sum_{i=0}^{\overline{k}} a_i a_j cov \left[\theta_t^{(i)}, \theta_t^{(j)}\right] \right\} \tag{6.17}$$

and denote its limit z_∞

$$z_\infty \equiv \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_i a_j cov \left[\theta_t^{(i)}, \theta_t^{(j)}\right] \tag{6.18}$$

If the limit exist we know that there exists a $\overline{k}$ such that

$$|z_\infty - z_{\overline{k}}| < \varepsilon : \varepsilon > 0 \tag{6.19}$$

From (6.16) the variance of the approximation error is given by

$$E\left(\Delta_{\overline{k},t}\right)^2 = \left(\mathbf{a} - \left[\begin{array}{cc} \mathbf{a}_{\overline{k}} & 0 \end{array}\right]\right) E\left(\theta_t^{(0:\infty)} \theta_t^{\prime(0:\infty)}\right) \left(\mathbf{a} - \left[\begin{array}{cc} \mathbf{a}_{\overline{k}} & 0 \end{array}\right]\right)' \tag{6.20}$$

$$= \sum_{j=\overline{k}+1}^{\infty} \sum_{i=\overline{k}+1}^{\infty} a_i a_j cov \left[\theta_t^{(i)}, \theta_t^{(j)}\right] \tag{6.21}$$

which equals the difference between $z_{\overline{k}}$ and its limit so that

$$z_\infty - z_{\overline{k}} = \sum_{j=\overline{k}+1}^{\infty} \sum_{i=\overline{k}+1}^{\infty} a_i a_j cov \left[\theta_t^{(i)}, \theta_t^{(j)}\right] \tag{6.22}$$

To prove that the approximation error tend to zero as $\overline{k}$ increases it is thus sufficient to show that the sequence $\{z_{\overline{k}}\}$ converges, i.e. that the limit z_∞ exists and is finite. Taking variances of both sides of (6.1) as $\overline{k} \to \infty$, we find that the variance of the equilibrium price is given by

$$\begin{aligned} E\left[p_t\right]^2 = & \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_i a_j cov \left[\theta_t^{(i)}, \theta_t^{(j)}\right] \\ & + 2\delta\gamma\lambda \sum_{j=0}^{\infty} a_j cov \left[\theta_t^{(j)}, \epsilon_t\right] \\ & + (\delta\gamma\lambda)^2 \sigma_\epsilon^2 + \left(\frac{\lambda\psi}{1-\lambda\psi}\right)^2 \sigma_c^2 \end{aligned} \tag{6.23}$$

By Lemma 3 we know that the variance of the price is finite so that $E\left[p_t\right]^2 < \infty$. This in turn implies that each term on the right hand side of (6.23) also must be finite. Since the first term on the right hand side of (6.23) equals the limit z_∞ we know that $\{z_{\overline{k}}\}$ converges. There thus exists a finite $\overline{k}$ such that

$$E\left(\Delta_{\overline{k}}\right)^2 = |z_\infty - z_{\overline{k}}| \tag{6.24}$$

$$< \varepsilon : 0 < \varepsilon \tag{6.25}$$

where the first line follows from (6.20) - (6.22) and the second line follows from (6.19) and the fact that $\{z_{\overline{k}}\}$ converges. □

The equilibrium dynamics of the model can thus be approximated to an arbitrary accuracy by a finite dimensional representation. In the section following the next, we demonstrate that "finite" can be a quite small number so that he results of this section are practically useful. But first, we turn to how an equilibrium representation can be found in practice.

7. A Solution Algorithm

Solving the model implies finding the matrices M and N in the law of motion for the hierarchy of expectations, the row vector $\mathbf{a}$ in the price equation and the conditional variance δ. This section presents an iterative solution algorithm and shows that a solution exists under general conditions using a version of Brouwer's fixed point theorem.

There are three basics steps in each iteration (indexed by s) of the algorithm (i) For given a given law of motion of the hierarchy of expectations described by M_s and N_s and a given conditional variance δ_s, find the row vector $\mathbf{a}_{\overline{k},s}$ that maps the hierarchy of expectations $\theta_t^{(0:\overline{k})}$ into the price p_t. (ii) Conditional on given values of $M_s, N_s, \mathbf{a}_{\overline{k},s}$ and δ_s find the new matrices M_{s+1} and N_{s+1} of the law of motion of the hierarchy of expectations. (iii) For given M_{s+1}, $N_{s+1}, \mathbf{a}_{\overline{k},s}$ and δ_s find the new conditional variance δ_{s+1}. These steps are described in detail below. From now on, all derivations pertain to a finite dimensional approximation of the equilibrium. To simplify notation, subscripts indicating that vectors and matrices are finite dimensional have been suppressed at instances where this will not cause confusion.

7.1. **Step 1: Computing the price.**

The first step is to find the price of the asset as a function of the contemporaneous expectation hierarchy of the supply disturbance $\theta_t^{(0:\overline{k})}$ for a given law of motion of the hierarchy and a given conditional variance of $(p_t + c_t)$, δ. That is, we want to find $\mathbf{a}_{\overline{k}}$ in

$$p_t = \mathbf{a}_{\overline{k}}\theta_t^{(0:\overline{k})} - \delta\gamma\lambda\epsilon_t + \frac{\lambda\psi}{1-\lambda\psi}c_t \tag{7.1}$$

as a function of M and δ. This simply entails computing the infinite sum in (5.6) so that

$$\mathbf{a}_{\overline{k}}\theta_t^{(0:\overline{k})} = -\delta\gamma\lambda\sum_{k=0}^{\infty} e_1'\left(\lambda MH\right)^k\theta_t^{(0:\overline{k})} \tag{7.2}$$

For a finite $\overline{k}$ and given M_s and δ_s implies that $\mathbf{a}_{\overline{k},s}$ is given by

$$\mathbf{a}_{\overline{k},s} = -\delta_s\gamma\lambda_s e_1'\left(I - \lambda_s M_s H\right)^{-1} \tag{7.3}$$

where we know that $\lambda_s M_s H$ is stable matrix (since the variance of p_t is finite). The price function thus resembles a standard discounted expected sum of future fundamentals, but where the coefficient matrix M from the true law of motion is replaced with MH.

7.2. **Step 2: The dynamics of the expectation hierarchy.** The state consists of the actual supply disturbance θ_t and the hierarchy of expectations of the supply disturbance $\theta_t^{(0:\overline{k})}$, so the law of motion of the state is determined by the actual supply process (3.6) and the law of motion of the higher order estimates. The Kalman filter thus plays a dual role: it both determines traders' estimate of the state as well as the law of motion of the very same state that the traders are estimating. To find the updated law of motion M_{s+1} and N_{s+1} we first derive the recursive updating equation for trader j's estimate of the hierarchy of higher order expectations conditional on the the previous law of motion M_s and N_s. We then take averages of this recursive updating equation to find the new law of motion for the hierarchy of average expectations M_{s+1} and N_{s+1}.As inputs, we will also need the vector $\mathbf{a}_{\overline{k},s}$ from the s iteration of the price function (7.3) and conditional variance δ_s.

7.2.1. *Trader j's estimate of the hierarchy.* For given values M_s and N_s, the conjectured law of motion for the hierarchy (3.18) and trader j's information set (3.12) can be written as a state space system of the form

$$\theta_t^{(0:\overline{k})} = M_s\theta_{t-1}^{(0:\overline{k})} + N_s w_t \tag{7.4}$$

$$S_t(j) = L_s\theta_t^{(0:\overline{k})} + Q_s c_t + \begin{bmatrix} R_{1s} & R_2 \end{bmatrix} \begin{bmatrix} w_t \\ w_t(j) \end{bmatrix} : \begin{bmatrix} w_t \\ w_t(j) \end{bmatrix} \sim N(0,I) \tag{7.5}$$

where the following definitions were used

$$S_t(j) = \begin{bmatrix} s_t(j) \\ p_t \end{bmatrix},\ L_s = \begin{bmatrix} e_1' \\ \mathbf{a}_{\overline{k},s} \end{bmatrix},\ Q = \begin{bmatrix} 0 \\ \frac{\lambda_s\psi}{1-\lambda_s\psi} \end{bmatrix}$$

$$R_{1s} = \begin{bmatrix} 0 & 0 \\ 0 & -\delta_s\gamma\lambda_s\sigma_\epsilon \end{bmatrix},\ R_2 = \begin{bmatrix} \sigma_\eta \\ 0 \end{bmatrix}$$

The subscript s indicates that a matrix may be changing at each iterative step in the algorithm. Trader j estimates the hierarchy of contemporaneous expectations recursively, using the Kalman filter updating equation

$$\theta_t^{(1:\overline{k})}(j) = M_s\theta_{t-1}^{(1:\overline{k})}(j) + K_s\left[S_t(j) - L_sM_s\theta_{t-1}^{(1:\overline{k})}(j) - Qc_t\right] \tag{7.6}$$

The Kalman gain K_s is given by

$$K_s = (P_sL_s' + N_sR_{s1}')(L_sP_sL_s' + R_sR_s')^{-1} \tag{7.7}$$

$$P_s = M_s\left(P_s - (P_sL_s' + N_sR_{1s}')(L_sP_sL_s' + R_sR_s')^{-1}(P_sL_s' + N_sR_{1s}')'\right)M_s' + N_sN_s' \tag{7.8}$$

and where $R_s = \begin{bmatrix} R_{1s} & R_2 \end{bmatrix}$. Note that the s subscript on K and P denotes the step in the solution algorithm, not the time period. That is, for given M_s and N_s, we compute the time invariant, or steady state, Kalman gain K_s.

7.2.2. *The average expectation hierarchy.* We want to find the conjectured vector AR(1) law of motion (3.18) for the hierarchy of average contemporaneous expectations, that is, we want to find the matrices M and N. We thus need to integrate the state updating equation (7.6) across traders and express all remaining terms as functions of the lagged expectation hierarchy $\theta_{t-1}^{(0:\overline{k})}$ and the aggregate shocks w_t. Use the definition of the private signal $s_t(j)$ (3.11), the price equation (3.15) and that $\int R_2 w_t(j)\, dj = \mathbf{0}$ to write the average signal $S_t \equiv \int S(j)\, dj$ as

$$S_t = L_s M_s \theta_{t-1}^{(0:\overline{k})} + L_s N_s w_t + R_{1s} w_t \tag{7.9}$$

Substituting the average signal (7.9) into the updating equation (7.6) gives the law of motion of the average of traders' estimate of the state

$$\theta_t^{(1:\overline{k})} = (I - K_s L_s)\, M_s \theta_{t-1}^{(1:\overline{k})} + K_s L_s M_s \theta_{t-1}^{(0:\overline{k})} + (K_s L_s N_s + K_s R_{1s})\, w_t \tag{7.10}$$

The final step to get the conjectured form (3.18) is to collect terms and append the actual supply disturbance process

$$\theta_t = \rho\theta_{t-1} + v_t \tag{7.11}$$

to the updating equation (7.10) to get

$$\begin{aligned} \begin{bmatrix} \theta_t \\ \theta_t^{(1:\overline{k})} \end{bmatrix} &= \begin{bmatrix} \rho & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \theta_{t-1} \\ \theta_{t-1}^{(1:\overline{k})} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ K_s L_s M_s \end{bmatrix} \begin{bmatrix} \theta_t \\ \theta_t^{(1:\overline{k})} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & (I - K_s L_s)\, M_s \end{bmatrix} \begin{bmatrix} \theta_{t-1} \\ \theta_{t-1}^{(1:\overline{k})} \end{bmatrix} + \begin{bmatrix} \sigma_v e_1' \\ (K_s L_s N_s + K R_1) \end{bmatrix} w_t \end{aligned} \tag{7.12}$$

where the last row and/or columns of the matrices have been cropped to make the matrices conformable (i.e. implementing the approximation that expectations of order $k > \overline{k}$ are redundant, and therefore setting $\theta_t^{(k)} = 0 : k > \overline{k}$). Equating coefficients in (7.12) and (3.18) then gives the updated matrices M_{s+1} and N_{s+1}

$$M_{s+1} = \begin{bmatrix} \rho & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ K_s L_s M_s \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & (I - K_s L_s)\, M_s \end{bmatrix} \tag{7.13}$$

$$N_{s+1} = \begin{bmatrix} \sigma_v e_1' \\ (K_s L_s N_s + K_s R_{1s}) \end{bmatrix} \tag{7.14}$$

7.3. **Step 3: The conditional variance.** The conditional variance of $(c_{t+1} + p_{t+1})$, δ, is the variance of investors' forecast error of the sum $c_{t+1} + p_{t+1}$ based on their period t information sets. The conditional forecast error is given by

$$\delta_{s+1} = \widehat{\mathbf{a}}_{\overline{k},s} \widehat{P}_s \widehat{\mathbf{a}}_{\overline{k},s}' + \left[1 + 2\frac{\lambda_s \psi}{1 - \lambda_s \psi} + \left(\frac{\lambda_s \psi}{1 - \lambda_s \psi} \right)^2 \right] \sigma_u^2 \tag{7.15}$$

$$\widehat{\mathbf{a}}_{\overline{k},s} = \begin{bmatrix} \mathbf{a}_{\overline{k},s} & -\delta_s \gamma \lambda_s \end{bmatrix} \tag{7.16}$$

where $\widehat{P}_s$ is the one period ahead joint forecast error covariance matrix of ϵ_t and the hierarchy of expectations of θ_t. Details on how to compute $\widehat{P}_s$ are given in the Appendix.

7.4. **The existence of a fixed point.** Solving the model implies finding a fixed point of equations (7.3),(7.7),(7.8),(7.13),(7.14) and (7.15). We now prove the existence of such a fixed point using Brouwer's fixed point theorem, which we first restate.[6]

[6] The relevant version of Brouwer's fixed point theorem is for compact subsets of $\mathbb{R}^n$ and is also known as *Kakutanis' fixed point theorem.*

Lemma 4. *(Brouwer fixed point theorem) Every continuous map from a convex compact set into itself has a fixed point.*

We thus need to show that iterating on Step 1 - 3 above is indeed a map from a convex compact set into itself. In order to do so, we will redefine the mapping $\{M_s, N_s, a_{\overline{k}}, \delta_s, K_s, L_s\} \rightarrow \left\{M_{s+1}, N_{s+1}, a_{\overline{k},s+1}, \delta_{s+1}, K_{s+1}, L_{s+1}\right\}$ described by Step 1 - 3 above in two ways.

First, note that for given M_{s+1}, N_{s+1} and δ_{s+1} we can find $a_{\overline{k},s+1}, K_{s+1}$ and L_{s+1} that do not depend on $a_{\overline{k},s}, K_s$ and L_s. It is thus sufficient to find a fixed point of the mapping $\{M_s, N_s, \delta_s\} \rightarrow \{M_{s+1}, N_{s+1}, \delta_{s+1}\}$.

Secondly, we will redefine the matrix M as an equivalent function of two covariance matrices with known properties, i.e. matrices that belong to the convex compact set $\mathcal{S}$, which we now define.

Definition 5. *The set $\mathcal{S}$ is the set of $\left(\overline{k}+1\right) \times \left(\overline{k}+1\right)$ matrices Σ matrix with i^{th}row, j^{th} column element $\sigma_{i,j}$ such that $|\sigma_{i,j}| \leq E\left(\theta_t^2\right) : i, j = 1, 2, ..., \overline{k}+1$*

Lemma 5. *The matrix M_s can equivalently be expressed as a function of the matrices Σ_s and $\Sigma_{+1,s}$ defined as*

$$\Sigma_s \equiv cov_s\left(\theta_t^{(0:\overline{k})}, \theta_t^{(0:\overline{k})}\right) \tag{7.17}$$

$$\Sigma_{+1,s} = cov_s\left(\theta_{t+1}^{(0:\overline{k})}, \theta_t^{(0:\overline{k})}\right) \tag{7.18}$$

where cov_s denotes the covariance conditional on the law of motion described by M_s and N_s.

Proof. From the projection theorem we know that

$$E\left[\theta_{t+1}^{(0:\overline{k})} \mid \theta_t^{(0:\overline{k})}\right] = \Sigma_{+1,s}\Sigma_s^{-1}\theta_t^{(0:\overline{k})} \tag{7.19}$$

i.e. M_s is given by $\Sigma_{+1,s}\Sigma_s^{-1}$. □

Lemma 6. *The covariance matrices Σ_s and $\Sigma_{+1,s}$ belong to $\mathcal{S}$, that is, that all elements of Σ_s and $\Sigma_{+1,s}$ lie in the closed interval $\left[-E\left(\theta_t^2\right), E\left(\theta_t^2\right)\right]$.*

Proof. The mapping $\{\Sigma_s, \Sigma_{+1,s}, N_s, \delta_s\} \rightarrow \{\Sigma_{s+1}, \Sigma_{+1,s+1}, N_{s+1}, \delta_{s+1}\}$ defines a new law of motion for the hierarchy $\theta_{t,s+1}^{(1:\overline{k})}$ that is the optimal estimate of the hierarchy $\theta_{t,s}^{(0:\overline{k})}$ if $\theta_{t,s}^{(0:\overline{k})}$ is governed by the law of motion $\{M_s, N_s\}$. We know that the variance of an optimal estimate cannot be larger than the variance of the object being estimated, so the inequality

$$E\left(\theta_{t,s}^{(k)}\right)^2 \geq E\left(\theta_{t,s+1}^{(k+1)}\right)^2 \tag{7.20}$$

must hold for each iteration s. Starting from an initial guess of M_0 and a N_0 (for instance the M and N implied by the full information solution) such that

$$E\left(\theta_t^2\right) \geq E\left(\theta_{t,0}^{(k)}\right)^2 : k = 1, 2, ... \tag{7.21}$$

ensures that

$$\Sigma_s \in \mathcal{S}, \Sigma_{+1,s} \in \mathcal{S} : s = 0, 1, 2, ... \tag{7.22}$$

The last result follows from the Cauchy-Schwarz inequality (in L^2 with the square root norm)

$$|E(XY)| \leq \sqrt{E(X)^2}\sqrt{E(Y)^2} \tag{7.23}$$

so that

$$\begin{aligned}\left|cov(\theta_{t+s}^{(k)},\theta_t^{(k+l)})\right| &\leq \max\left\{E\left(\theta_t^{(k)}\right)^2, E\left(\theta_{t+s}^{(k+l)}\right)^2\right\} : k,l,s=0,1,2... &(7.24)\\ &\leq E\left(\theta_t^2\right) &(7.25)\end{aligned}$$

i.e. all elements of Σ_s and $\Sigma_{+1,s}$ must lie in the closed interval $[-E\left(\theta_t^2\right), E\left(\theta_t^2\right)]$ which proves the lemma. □

Definition 6. *The set $\mathcal{N}$ is the set such that if $N \in \mathcal{N}$ then N is $\left(\overline{k}+1\right)\times 2$ matrix with elements $|n_{i,j}| \leq \sqrt{E\left(\theta_t^2\right)} : i = 1,2,...,\overline{k}+1$ and $j = 1,2$.*

Lemma 7. *The matrices $N_s : s = 0,1,2...$ in the iteration described by $\{\Sigma_s, \Sigma_{+1,s}, N_s, \delta_s\} \rightarrow \{\Sigma_{s+1}, \Sigma_{+1,s+1}, N_{s+1}, \delta_{s+1}\}$ belong to $\mathcal{N}$.*

Proof. $M_s\Sigma_sM_s'$ in the Lyaponov equation for Σ_s

$$\Sigma_s = M_s\Sigma_sM_s' + N_sN_s' \tag{7.26}$$

is a positive semi-definite matrix. Since $\Sigma_s \in \mathcal{S}$ for each iteration s, each element $n_{s,ij}$ of $N_s$ust lie in the interval $[-\sqrt{E\left(\theta_t^2\right)}, \sqrt{E\left(\theta_t^2\right)}]$ since the i^{th} element on the diagonal of N_sN_s' is given by

$$\left(N_sN_s'\right)_{ii} = \sum_{j=1}^{\overline{k}+1} n_{ij}n_{ij} \tag{7.27}$$

The results then follows immediately the fact that the diagonal elements of Σ_s are non-negative for positive semi-definite matrices. □

Definition 7. *The set $\mathcal{D}$ is the closed interval $[0, \overline{\sigma}_{pc}^2]$ on $\mathbb{R}$ where $\overline{\sigma}_{pc}^2$ is the upper bound of the unconditional variance of $p_t + c_t$.*

We then have that $\delta_s \in \mathcal{D}$ since the conditional variance is bounded by the unconditional variance

$$E\left(p_t + c_t\right)^2 \leq E\left(p_t\right)^2 + E\left(c_t\right)^2 + 2\max\left\{E\left(p_t\right)^2, E\left(c_t\right)^2\right\} \tag{7.28}$$

where the inequality follows from the Cauchy-Schwarz inequality and that both the price and coupon payments have finite variances.

Proposition 6. *The set $\mathcal{Z} \equiv \mathcal{S} \times \mathcal{S} \times \mathcal{N} \times \mathcal{D}$ is convex and compact and a fixed point described by the iteration $\{\Sigma_s^{-1}, \Sigma_{+1,s}, N_s, \delta_s\} \rightarrow \left\{\Sigma_{s+1}^{-1}, \Sigma_{+1,s+1}, N_{s+1}, \delta_{s+1}\right\}$ exists.*

Proof. For finite dimensional sets, compactness is equivalent to a set being closed and bounded, so compactness follows directly from the definitions of $\mathcal{S}$, $\mathcal{N}$ and $\mathcal{D}$. Convexity follows from that if $|x| \leq c$ and $|z| \leq c$ then $\alpha|x| + (1-\alpha)|z| \leq c$. The existence of a fixed point then follows from Lemma 4 - 7. □

In this section we have shown how a solution to the model can be found for a finite $\overline{k}$. In practice, we need to choose a maximum order of expectations to include in the representation of the model. The next section shows how this can be done by ensuring that the impulse responses function for prices have converged and thus remain unchanged as the maximum order $\overline{k}$ is increased further.

8. Properties of the Solved Model

In this section, the properties of the model is explored in more detail. First, we give an example of how the private information changes the price responses to supply shocks in contrast to when the model is solved under full or imperfect but common information. Secondly, we demonstrate how the representation of the equilibrium dynamics of the model can be used to compute two different types of dispersion of expectations: (i) Dispersion of conditional expected returns across traders, and dispersion across orders of expectations. Both of these types of dispersion may be of interest to quantify and it is straight forward to compute either for a given parameterization of the model.

8.1. **Dynamics.** One question of interest is how private information affect the responses of the asset's price to innovations to the supply of assets. In the top row of Figure 1 below, we have plotted the impulse response function of the price of the asset to an innovation to the persistent component of supply (left column) and to a transitory shock (right column). For comparison, we have also plotted the impulse response to the same innovation under the alternative assumptions of full information, i.e. the state is observed perfectly by all traders, and imperfect but common information, i.e. it is common knowledge that all traders observe the same noisy signal about θ_t. The parameters $\left\{\overline{k}, \gamma, \xi, \psi, \rho, \overline{r}, \sigma_u^2, \sigma_v^2, \sigma_\epsilon^2, \sigma_{\eta i}^2, \right\}$ was set to $\{15, 1, 1.5, 0.5, 0.9, 0.01, 0.01, .1, 0.001, 1\}$. For the imperfect but common information case we set the variance of the noise in the common signal to the same as the idiosyncratic noise variance in the private information case.

The impulse response functions for this parameterization are displayed in Figure 1 which demonstrates that the different information structures imply very different price dynamics. Both private and common imperfect information results in weaker initial responses to a persistent supply shock compared to the full information case, with the trough appearing later with private information than with imperfect but common information. Imperfect information also makes the price response to a transitory shock persistent and the persistence is stronger with private signals than with an equally precise common signal. [7]

That private information can be a strong force of inertia in endogenous variables has been noted before, e.g. Woodford (2002), Nimark (2008), Graham and Wright (2010) and Angeletos and L'ao (2009). As first pointed out by Woodford (2002) in a setting where agents faced a dynamic filtering problem (but with static choices), it is the fact that higher order expectations respond much more sluggishly to a shock than lower order expectations that

[7] Singleton concluded that what mattered most in his model was that agents had *imperfect* information, rather than *private* information per se. An earlier version of this paper demonstrated that this was due to the large variances of the innovations in the supply process in Singleton's calibration. Since the discount factor λ depends on the conditional variance of returns δ, absolute (and not only relative) variances of shocks matter. Larger variances imply faster discounting of the higher order expectation in (3.15).

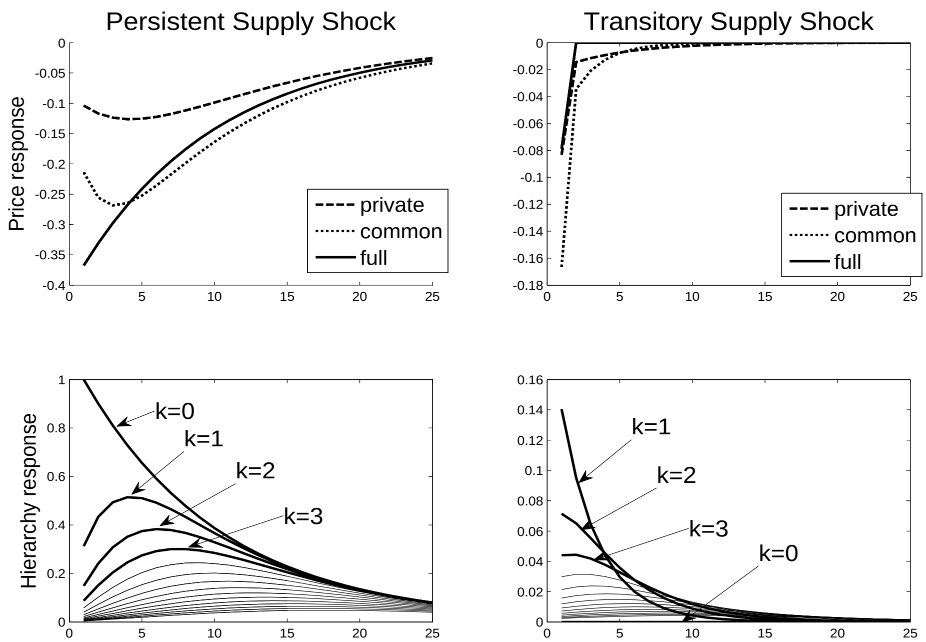

FIGURE 1. Impulse responses of p_t (top row) and $\theta_t^{(0:50)}$ (bottom row) to innovation to persistent (left column) and transitory (right column) component of supply.

causes the inertial response of the endogenous variable. This is illustrated in the bottom row of Figure 1 where the responses of the hierarchy of expectations to the two shocks are plotted. Average first order expectations ($k = 1$) respond stronger than higher order expectations in the impact period to both persistent and transitory shocks. That higher order expectation respond less than lower order expectations is intuitive. First order expectation respond less than the true shock on impact since some of the actual supply shock will be attributed to the transitory shock. Since traders know that first order expectations on average respond less than the actual shock, second order expectations must respond less than first order shock. This argument can then be applied recursively to understand why a $k+1$ order expectation responds less than a k order expectation in the impact period.

After a transitory shock ϵ_t and for $k \geq 1$, lower order expectations of θ_t also respond more strongly on impact. However, lower order expectations respond quicker to the higher than expected asset prices that follows the impact period and converge faster towards the true shock (zero) than lower order expectations. The fact that convergence of (higher order) expectations about θ_t towards zero is not immediate introduces some persistence of the price response also to a transitory shock.

8.2. **Cross-sectional dispersion of expectations.** Survey evidence suggest that market participants may have dispersed expectations about future economic outcomes, e.g. Swanson (2006) and Mankiw, Reis and Wolfers (2003). Private information is one way of introducing such dispersion in a model and there are at least two reason why this may be of interest. First, we may want to use quantitative information from for instance surveys to calibrate the parameters of a model to match the measured dispersion of expectations. Secondly, and as in Nimark (2010), computing the implied dispersion for a model with parameters estimated using only aggregate variables, one can gauge the plausibility of the model by judging whether the dispersion of expectations implied by the parameters that generate the best fit to aggregate variables is realistic. In the framework presented here, it is straight forward to compute the cross-sectional dispersion of expectations.

The idiosyncratic noise shocks $\eta_t(j)$ are white noise processes that are orthogonal across traders and to the aggregate shocks v_t and ϵ_t. This implies that the cross-sectional variance of expectations is equal to the part of the unconditional variance of trader j's expectations that is due to idiosyncratic shocks. This quantity can be computed by finding the variance of the estimates in trader j's updating equation (7.6), but with the aggregate shocks v_t and ϵ_t "switched off". The covariance Σ_j of trader j's state estimate due to idiosyncratic shocks is defined as

$$\Sigma_j \equiv E\left(\theta_t^{(1:\bar{k})}(j) - \int \theta_t^{(1:\bar{k})}(j')dj'\right)\left(\theta_t^{(1:\bar{k})}(j) - \int \theta_t^{(1:\bar{k})}(j')dj'\right)' \tag{8.1}$$

and given by the solution to the Lyaponov equation

$$\Sigma_j = (I - KL)\, M\Sigma_j M' \,(I - KL)' + KR_2R_2'K'. \tag{8.2}$$

The cross-sectional dispersion of expectations about endogenous variables are caused by cross-sectional dispersion of expectations about the state. Agent j's expectation of the price s periods ahead is given by

$$E\left[p_{t+s} \mid \Omega_t(j)\right] = \mathbf{a}_{\bar{k}} M^s E\left[\theta_t^{(0:\bar{k})} \mid \Omega_t(j)\right] \tag{8.3}$$

so that the cross-sectional price expectation dispersion can be computed as

$$E\left(E\left[p_{t+s} \mid \Omega_t(j)\right] - \int E\left[p_{t+s} \mid \Omega_t(j')\right] dj'\right)^2 = \mathbf{a}_{\bar{k}} M^s \Sigma_j \left(\mathbf{a}_{\bar{k}} M^s\right)' \tag{8.4}$$

The cross sectional variance of expectations will generally depend on all the parameters of the model, but some have a more direct influence on the dispersion than others. For instance, Figure 2 illustrates how the cross sectional variance of one period ahead price expectations depends on the variance σ_η^2 of the idiosyncratic noise shock $\eta_t(j)$ (left panel) and the variance σ_ϵ^2 of the transitory demand shock ϵ_t (right panel). Both graphs start at the origin, i.e. if either the variance of the idiosyncratic noise shocks or the transitory supply shocks are zero, there is no dispersion of expectations. Of course, if there are no idiosyncratic noise shocks, there is no private information since all traders observe θ_t directly and without error. Similarly, if there are no transitory supply shocks, traders can infer θ_t perfectly from observing the price p_t and again, there is no private information in equilibrium. This result is reminiscent of the result in Walker (2007) who uses a version of Singleton's model to show

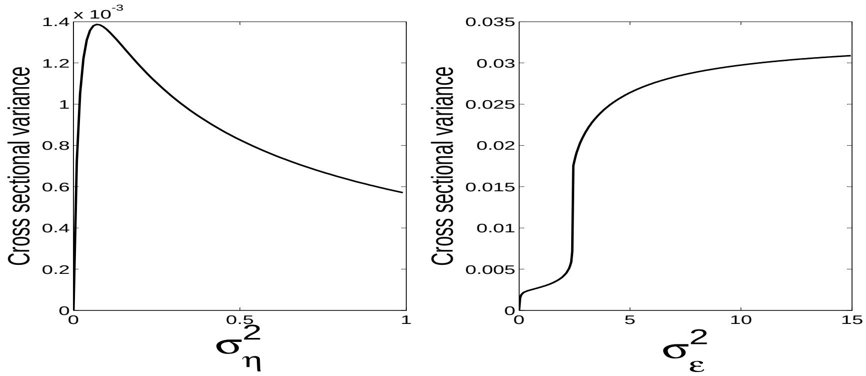

FIGURE 2. Cross sectional variance of one period ahead price expectation.

that if one of the supply shocks is observed directly, equilibrium prices reveal the other shock perfectly and there is then no role for private information.

While the limit case of zero variance is similar for the two shocks in the figure, the change in dispersion as the variance is increased is quite different. If the variance of the idiosyncratic shock is increased, dispersion of expectations first increases as traders observe private signals with a larger cross-sectional dispersion. However, at some point the variance of the idiosyncratic noise shocks become large enough so that the weight on the private signal decreases faster than the variance of the noise increases. This explains the hump shape dependence of cross-sectional dispersion on the variance σ_η^2.

We do not see the hump shape in the right panel. The reason is that when the variance of the transitory shock is increased, prices become more noisy as signals about θ_t and traders tend to put more weight on their private signal $s_t(j)$. Where the graph flattens out, the price is so noisy that traders do not put any weight on it at all when estimating θ_t.

8.3. **Dispersion across orders of expectations.** The framework presented here can also be used to compute a different type of dispersion of expectations, that is, when different orders of expectations do not coincide. Unlike the cross-sectional dispersion, dispersion across orders of expectations vary over time and gives rise to new dynamics. Indeed, it is the fact that there is a divergence between orders of expectations that makes models with private information to display different dynamics since the full information rational expectations equilibrium can be thought of as a special case when all orders of expectations coincide in every period so that $\theta_t = \theta_t^{(k)} : k = 1, 2, ...$for all t. As with the cross-sectional dispersion, the amount of dispersion across orders of expectations generally depends on all the parameters in the model but the variance of the transitory supply shock and the variance of the idiosyncratic noise shock again play a more direct role. Figure 3 illustrates how the response of the hierarchy of expectations of θ_t from order zero to 50 to a unit innovation in θ_t depend on the variance of the transitory supply shock ϵ_t. (Apart from σ_ϵ^2, the parameterization is the same as that used for Figure 1.) The thick solid line is the

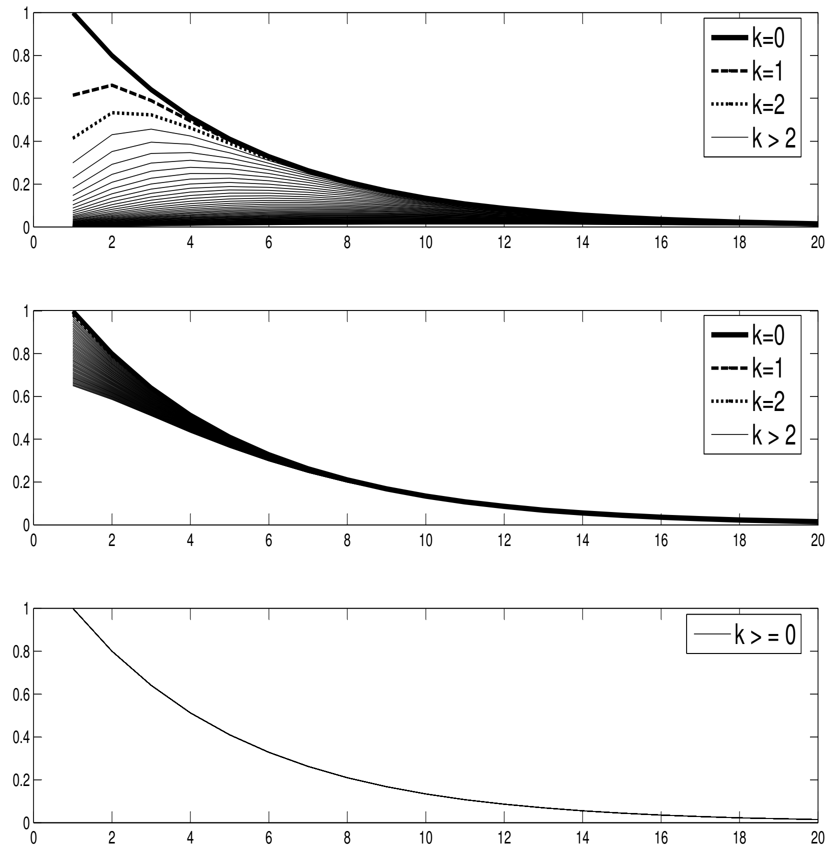

FIGURE 3. Impulse responses of $\theta_{t|t}^{(0:50)}$ to a unit innovation v_t

response of the actual shock, or $\theta_t^{(0)}$, the dashed line immediately beneath it is the first order expectations, the dotted line next is the second order expectation and so on. The transitory supply shock ϵ_t functions as aggregate noise that prevents the price from perfectly revealing θ_t. If we decrease its variance, equilibrium prices will be more informative about θ_t and other traders' (higher order) expectations of θ_t. This can be seen in the mid panel of Figure 5, where we have plotted a second impulse response function for the hierarchy $\theta_t^{(0:50)}$. The variance of ϵ_t in the middle panel is set to 1/10 of that in the top panel. It is clear that decreasing the variance of the transitory shock makes all orders of expectations move closer together, i.e. making traders better informed about all orders of expectations of θ_t.

From a filtering perspective, setting the variance of ϵ_t equal to zero is equivalent to making it perfectly observable. The bottom panel demonstrates that the model with $\sigma_\epsilon^2 = 0$ replicates the result of Walker (2007): Equilibrium prices perfectly reveal the value of θ_t so that all orders of expectations coincide and the graph collapses to a single line. However, this is not a general property of Singleton's model, but an artefact of the additional assumptions that $\sigma_\epsilon^2 = 0$, or equivalently, that traders can observe ϵ_t perfectly.

8.4. **Accuracy.** In the previous section, we demonstrated that a finite number of orders of expectations are sufficient to accurately represent the equilibrium dynamics of the model. In practice, a maximum order $\overline{k}$ need to be chosen such that we are confident that including a larger number of orders of expectations would not change the dynamics of the model. Here, we illustrate that for both the row vector $\mathbf{a}_{\overline{k}}$ and the impulse response functions to the aggregate shocks to converge, relatively few orders of expectations are needed.

In Figure 4, the row vector $\mathbf{a}_{\overline{k}}$ is plotted for $\overline{k} = 1, 3, ...10$. We can see that the vector converges quite rapidly. When 6 or 7 orders of expectations are included, adding higher order expectations beyond that does not further alter the elements of $\mathbf{a}_{\overline{k}}$. We can also see that the elements of $\mathbf{a}_{\overline{k}}$ converges quite rapidly towards zero, so that Proposition 6, which stated that the series $\{\Sigma a_k\}_{k=0}^{\infty}$ converges , seems to "bite" already for relatively low values of k.

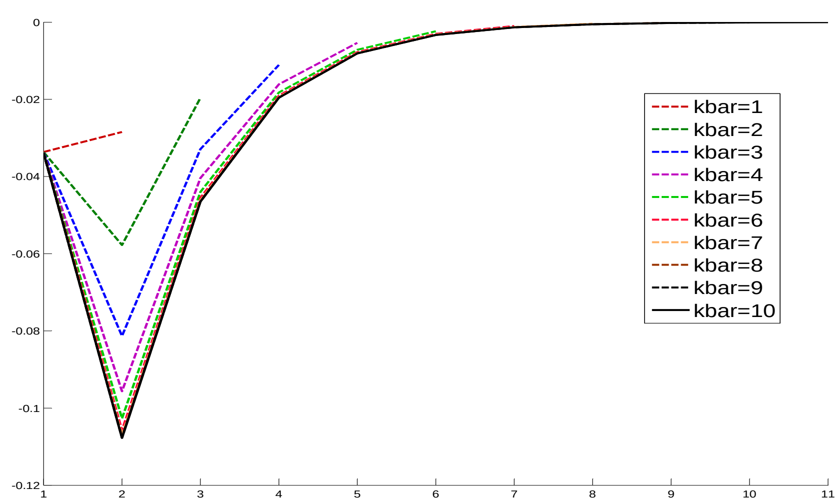

FIGURE 4. Equilibrium impact of k order (x-axis) expectation on price, i.e. the elements of $\mathbf{a}_{\overline{k}}$ for $\overline{k} = 1, 2, ..., 10$.

In order to have a satisfactory approximation to the infinite dimensional dynamics, we would also like the response of the endogenous price to aggregate shocks to converge. In Figure 5, the impulse response functions to persistent (left panel) and transitory (right panel) supply shocks are plotted for $\overline{k} = 1, 3, ...10$.

The impulse response functions of the price to the two aggregate shocks completely describes the endogenous dynamics of the model and they appear to converge rapidly. Visual

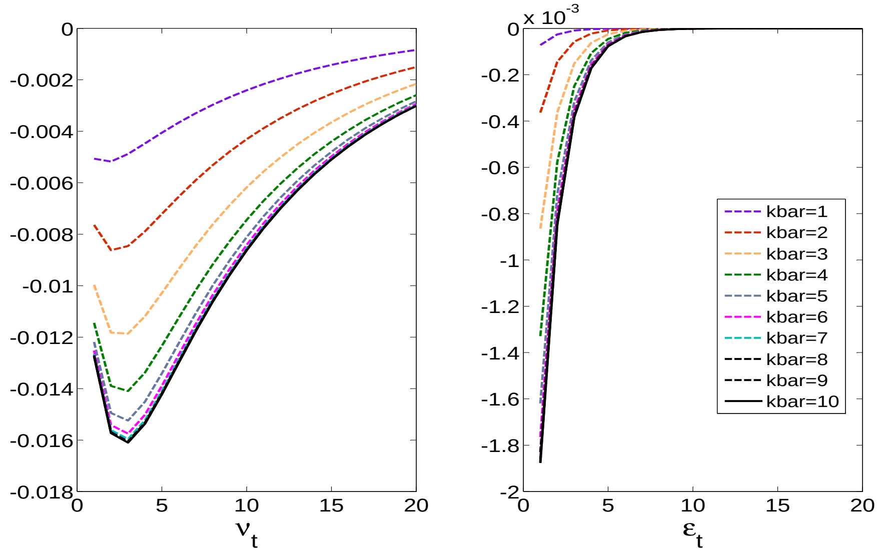

FIGURE 5. Impulse responses of p_t for $\overline{k} = 1, 2, ..., 10$

inspection of Figure 5 suggests that six or seven orders of expectations appear to be sufficient to accurately represent the equilibrium dynamics of the price. Of course, the number of orders of expectations required for an accurate solution depends on the parameters of the model. In general, the more persistent the supply shocks are, i.e. the closer ρ is to unity, the more orders of expectations are necessary. Also, the required number of orders of expectations has a maximum for intermediate levels of signal precision. For very precise signals higher order expectations about the future can be accurately captured by first order expectations, since there is then little dispersion across orders of expectations. At the opposite extreme, with very imprecise signals, higher order expectations do not respond much to shocks and they therefore then have little impact on price dynamics.

9. An equivalent alternative representation

Above, we showed that the equilibrium dynamics of the model could be approximated to an arbitrary accuracy by a finite dimensional representation. The approximation is implemented by truncating the state at a maximum order of expectation which we called $\overline{k}$. An alterative approach to find a finite dimensional representation builds on Townsend (1983) who proposed to solve a model in this class by assuming that the state is revealed perfectly with a lag. This approach avoids explicitly modeling higher order expectations by using that agents do not intrinsically care about the expectations of others, but about the actions of others. In a linear-Gaussian setting, these actions can be predicted directly using projection methods but it is generally optimal to condition on the entire history of observables. As time passes, the dimension of the vector of observables thus increases without bound. By assuming that the true state is revealed with a lag, the method effectively truncates the number of observables relevant for predicting the actions of other agents.

9.1. **Perfect state revelation with a (long) lag.** Versions of Townsend's method have been developed further by by Bacchetta and van Wincoop (2006), Hellwig (2002), Hellwig and Venkateswaran (2009). Hellwig (2002) and Hellwig and Venkateswaran (2009) assume that the state in period $t-T$ is revealed in period t where T can be a very large number. Intuitively, it seems plausible to conjecture that in a stationary environment, the equilibrium dynamics found using these methods should converge to some limit as T tends to infinity. Here we show formally that there does indeed exists a finite dimensional representation of the form proposed by Hellwig and Venkateswaran (2009) that as T tends to infinity converges to the same form as the representation derived above. In effect, they derive an equilibrium representation that is the sum of a finite order MA process plus a linear function of the perfectly revealed lagged state. Here we show that the representation derived in Section 6 and 7 can be rewritten in this form as the lag T tends to infinity.

Start by rewriting the law of motion (4.24) for the hierarchy $\theta_t^{(0:\overline{k})}$ in MA form

$$\theta_t^{(0:\overline{k})} = \sum_{s=0}^{\infty} M^s N w_{t-s} \tag{9.1}$$

The price (6.1) then has an alternative representation as the sum of an MA process in the supply shocks v_t and ϵ_t and a linear function of the perfectly known coupon payment c_t

$$p_t = \mathbf{a}_{\overline{k}} \sum_{s=0}^{\infty} M^s N w_{t-s} - \delta\gamma\lambda\epsilon_t + \lambda\psi c_t \tag{9.2}$$

We will now show that there exists a representation of the form used by Hellwig (2002) and Hellwig and Venkateswaran (2009) that as the lag T increases converges to (9.2). In those papers, projection methods are used to find the MA coefficients A_s in a solution of the form

$$p_t = \sum_{s=0}^{T} A_s w_{t-s} - \delta\gamma\lambda\epsilon_t + \widehat{p}_t \tag{9.3}$$

where $\widehat{p}_t$ is the "common knowledge" component of the current price which for the Singleton (1987) model is given by

$$\widehat{p}_t \equiv \frac{\rho^T}{1-\rho}\theta_{t-T} + \lambda\psi c_t. \tag{9.4}$$

As T tend to infinity, this representation converges to the form (9.2) since

$$\lim_{T\to\infty} \frac{\rho^T}{1-\rho}\theta_{t-T} = 0 \tag{9.5}$$

For a finite T there thus exists representation of the form (9.3) with MA coefficients given by $A_s = \mathbf{a}_{\overline{k}} M^s N$ that is arbitrarily close to the representation (9.2).

10. Conclusions

In this paper we derive a method for solving dynamic models with private information. The principal difficulty of solving models in this class is the infinite regress of expectations arising from agents' need to 'forecast the forecasts of others'. Here, we demonstrate how the infinite regress problem can be made tractable by imposing some structure on expectations. Specifically, it is common knowledge that agents form expectations rationally. This assumption allows us to derive the dynamics of higher order expectations explicitly and transparently.

We use the structure imposed on expectations by common knowledge of rationality to solve a version of Singleton's (1987) asset pricing model with privately informed traders. By defining an *average expectations operator*, we derive an expression for the price of the asset as a geometric sum that resembles the present discounted value of expected future fundamentals. While the functional form is similar to the corresponding expression in a full information model, there is an important difference since the price function is not derived by relying on the law of iterated expectations. Instead, the operator is used to compute a convergent sequence of higher order expectations of future fundamentals. The current price of the asset is given by the discounted sum of this sequence.

Determining the dynamics of higher order expectations and how these map into the price of an asset does not by itself solve the infinite regress problem. However, it does provide us with a framework that is tractable enough to derive conditions under which the model can be approximated to an arbitrary accuracy by a finite dimensional state representation. Incidentally, this is the same condition that guarantees that a stable solution exists in the full (or common) information case: If the discount rate multiplied by the eigenvalue of the fundamental process is smaller than unity in absolute value, we only need to model a finite number of orders of expectations to achieve any required degree of accuracy.

The equilibrium representation derived here can be taken as a literal description of agents' behavior, i.e. as representing agents who explicitly form expectations about other agents' expectations. The convergence results derived here can then be comforting for readers who find it implausible on cognitive limitations grounds that traders form an infinite number of higher order expectations. Indeed, what has been shown here is that forming only a finite and even low number of orders of expectations may in some settings be sufficient. An alternative interpretation is to view the equilibrium representation simply as a convenient functional form to model agents who have access to to private information and condition on the entire history of observables. The main advantage of the method is then to deliver a finite dimensional and time invariant representation of equilibrium dynamics.

While the model used to illustrate the method here had a scalar process as the latent fundamental, none of the proofs rely on this fact. The method also works well for a general vector valued latent process and have been applied both to calibrated macro models, as in Nimark (2008) and Graham and Wright (2010) as well as to estimated finance and macro models as in Nimark (2010) and Melosi (2011).

The literature has to date produced a wealth of qualitative results derived from the interactions that arise between agents when individuals have access to private information about variables of common interest. A natural next step is to test whether these qualitative results hold up when subjected to quantitative scrutiny. The solution method proposed in this paper allows us to solve dynamic models with private information accurately (and quickly) without making some of the modeling compromises previously thought to be necessary. In addition, the method delivers the solved model in a form that can be estimated directly by maximum likelihood methods. This paper helps shorten the step from qualitative to quantitative results by opening up the possibility of using dynamic models with privately informed agents that are realistic enough to use for empirical work.

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Appendix A. Computing the conditional variance

The conditional variance of $(c_{t+1}+p_{t+1})$, δ, is the variance of investors' forecast error of the sum $c_{t+1}+p_{t+1}$ based on their information sets in period t and is given by

$$\delta = E\left[\left(1+\frac{\lambda\psi}{1-\lambda\psi}\right)u_t + \mathbf{a}\theta_{t|t}^{(0:\overline{k})} - \mathbf{a}M\theta_{t-1|t-1}^{(1:\overline{k})} - \delta\gamma\lambda\epsilon_t\right]^2 \tag{A.1}$$

which can be rearranged to

$$\begin{aligned} \delta &= \left[1+2\frac{\lambda\psi}{1-\lambda\psi}+\left(\frac{\lambda\psi}{1-\lambda\psi}\right)^2\right]\sigma_u^2 \qquad \text{(A.2)} \\ &+\mathbf{a}P\mathbf{a}' + (\delta\gamma\lambda)^2\sigma_\epsilon^2 - 2E\left[\left(\mathbf{a}\theta_{t|t}^{(0:\overline{k})} - \mathbf{a}M\theta_{t-1|t-1}^{(1:\overline{k})}\right)\delta\gamma\lambda\epsilon_t\right] \end{aligned}$$

The expression on the second line of (A.2) can be computed by putting the hierarchy of contemporaneous expectations into state space form together with the transitory supply shock ϵ_t

$$\begin{bmatrix} \theta_{t|t}^{(0:\overline{k})} \\ \epsilon_t \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \theta_{t-1|t-1}^{(0:\overline{k})} \\ \epsilon_{t-1} \end{bmatrix} + \begin{bmatrix} N_1 & N_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix} \tag{A.3}$$

$$\begin{bmatrix} s_t(j) \\ p_t - \frac{\lambda\psi}{1-\lambda\psi}c_t \end{bmatrix} = \begin{bmatrix} e_1' & 0 \\ \mathbf{a} & -\delta\gamma\lambda \end{bmatrix}\begin{bmatrix} \theta_{t|t}^{(0:\overline{k})} \\ \epsilon_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\eta_t(j) \tag{A.4}$$

Define

$$X_t \equiv \begin{bmatrix} \theta_{t|t}^{(0:\overline{k})} \\ \epsilon_t \end{bmatrix} \tag{A.5}$$

$$\widehat{P} \equiv E\left(X_t - X_{t|t-1}\right)\left(X_t - X_{t|t-1}\right)' \tag{A.6}$$

$$\widehat{\mathbf{a}} \equiv \begin{bmatrix} \mathbf{a}_{\overline{k}} & -\delta\gamma\lambda \end{bmatrix} \tag{A.7}$$

then

$$\widehat{\mathbf{a}}\widehat{P}\widehat{\mathbf{a}}' = \mathbf{a}P\mathbf{a}' + (\delta\gamma\lambda)^2\sigma_\epsilon^2 - 2E\left[\left(\mathbf{a}_{\overline{k}}\theta_{t|t}^{(0:\overline{k})} - \mathbf{a}_{\overline{k}}M\theta_{t-1|t-1}^{(1:\overline{k})}\right)\delta\gamma\lambda\epsilon_t\right] \tag{A.8}$$

where $\widehat{P}$ is the one period ahead forecast error covariance matrix associated with the state space system (A.3)-(A.4). The conditional variance of the sum of the coupon payment and the price is then given by

$$\delta = \widehat{\mathbf{a}}\widehat{P}\widehat{\mathbf{a}}' + \left[1+2\frac{\lambda\psi}{1-\lambda\psi}+\left(\frac{\lambda\psi}{1-\lambda\psi}\right)^2\right]\sigma_u^2. \tag{A.9}$$

Appendix B. Proof of Lemma 4

Lemma 8. *The variance of the price p_t is finite*

Proof. We want to show that $E(p_t)^2 < \infty$. Taking variances of both sides of the expression for the equilibrium price (3.15) we get

$$\begin{aligned} E(p_t)^2 &= (\delta\gamma\lambda)^2 \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \lambda^{(i+j)} cov\left[\theta_{t+i|\cdots|t}, \theta_{t+j|\cdots|t}\right] && \text{(B.1)} \\ &\quad +2\delta\gamma\lambda \sum_{j=0}^{\infty} \lambda^j cov\left[\theta_{t+i|\cdots|t}, \epsilon_t\right] && \text{(B.2)} \\ &\quad + (\delta\gamma\lambda)^2 \sigma_\epsilon^2 + \left(\frac{\lambda\psi}{1-\lambda\psi}\right)^2 \sigma_c^2 && \text{(B.3)} \end{aligned}$$

The two terms on the last line are finite and given exogenously. We thus need to show that the infinite sums on the first and second line converge. We will do this by demonstrating that the absolute values of the covariance term is bounded by the variance of the true supply process, i.e

$$\left|cov\left[\theta_{t+i|\cdots|t}, \theta_{t+j|\cdots|t}\right]\right| \leq E(\theta_t)^2 \tag{B.4}$$

By the Cauchy-Schwartz inequality we know that

$$\left|cov\left[\theta_{t+i|\cdots|t}, \theta_{t+j|\cdots|t}\right]\right| \leq \max\left\{E\left(\theta_{t+i|\cdots|t}\right)^2, E\left(\theta_{t+j|\cdots|t}\right)^2\right\} \tag{B.5}$$

and from Proposition 2 we know that

$$E\left(\theta_{t+i|\cdots|t}\right)^2 \leq E(\theta_t)^2 \tag{B.6}$$

i.e. that the variance of higher order expectations are bounded by the variance of the true process. Applying these results to the first infinite series in (B.1) we have that

$$\begin{aligned} (\delta\gamma\lambda)^2 \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \left|\lambda^{(i+j)} cov\left[\theta^{(i)}_{t+i|\cdots|t}, \theta^{(j)}_{t+j|\cdots|t}\right]\right| &\leq (\delta\gamma\lambda)^2 \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \left|\lambda^{(i+j)} E(\theta_t)^2\right| && \text{(B.7)} \\ &= \frac{(\delta\gamma\lambda)^2}{(1-\lambda)^2} E(\theta_t)^2 && \text{(B.8)} \\ &< \infty && \text{(B.9)} \end{aligned}$$

Similarly, for the second infinite series, we have that

$$\begin{aligned} 2\delta\gamma\lambda \sum_{j=0}^{\infty} \lambda^j \left|cov\left[\theta_{t+i|\cdots|t}, \epsilon_t\right]\right| &\leq \max\left\{2\delta\gamma\lambda \sum_{j=0}^{\infty} \lambda^j E(\theta_t)^2, 2\delta\gamma\lambda \sum_{j=0}^{\infty} \lambda^j E(\epsilon_t)^2\right\} && \text{(B.10)} \\ &< \infty && \text{(B.11)} \end{aligned}$$

□