Factor models

May 23, 2011

Factor Models

- Macro economists have a peculiar data situation:
  - Many data series, but usually short samples

How can we utilize all this information without running into degrees of freedom problems?

- Factor models is one approach.

Today: Principal components and FAVARs

- Based on Stock and Watson (2010) and Bernanke, Boivin and Eliasz (2005).

Factor models

Dynamic factor model split data into a low dimensional dynamic component and a transitory series specific (idiosyncratic) component

Underlying assumption:

- A few common so called *factors* can explain most of the variation in many different time series

Factor models is one approach to so-called "dimension reduction" that sometimes help in estimation (and forecasting)

State space models

Factor models are a special case of state space models

$$\underset{(r\times 1)}{F_t} = AF_{t-1} + u_t$$
$$\underset{(N\times 1)}{Y_t} = \underset{(N\times r)}{W}\underset{(r\times 1)}{F_t} + v_t$$

where $N >> r$

Factor models

What is the interpretation of the factors?

- In most cases, there is no interpretation: The factors are statistical constructs that have no deeper meaning beyond their definitions
- In some context the factors may correspond to the some combination of state variables in a DSGE model (but one needs to be careful)

This has not stopped people from labeling factors:

- Real and nominal factors (Ng and Ludvigson JoF 2009)
- Level, slope and curvature (Large literature on bond yields)

When does it work?

Can we from just observing Y_t tell whether a large data set can be represented with a factor structure?

$$\begin{aligned} \underset{(r\times 1)}{F_t} &= AF_{t-1} + u_t \\ \underset{(N\times 1)}{Y_t} &= \underset{(N\times r)}{W}\underset{(r\times 1)}{F_t} + v_t \end{aligned}$$

Yes:

- Scree plots (informal but useful)

Scree plots

Uses that $Ev_t v_t' << EY_t Y_t'$ implies a particular structure of $EY_t Y_t'$ if $N >> r$

- Do eigenvalue-eigenvector decomposition of (normalized) sample covariance $EY_t Y_t'$

$$EY_t Y_t' = W \Lambda W'$$

where W contains the eigenvectors of $EY_t Y_t'$ and Λ is a diagonal matrix containing the ordered eigenvalues and where the eigenvectors are orthonormal so that $WW' = I$.

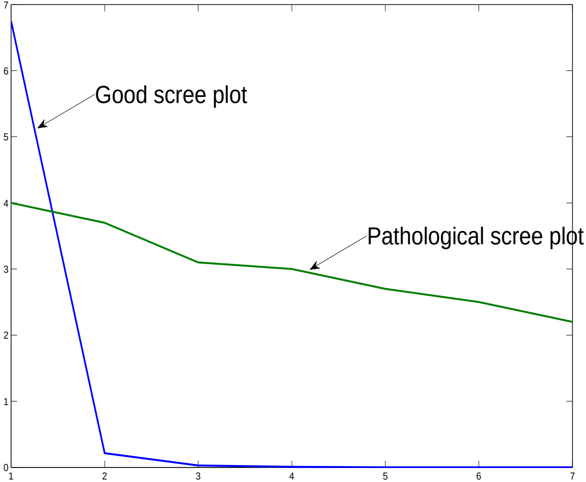

Figure: Good and bad scree plot

An example: Bond yields

The data:

Use Fed Funds Rate, 3, 12, 24, 36, 48, 60 month bond yields

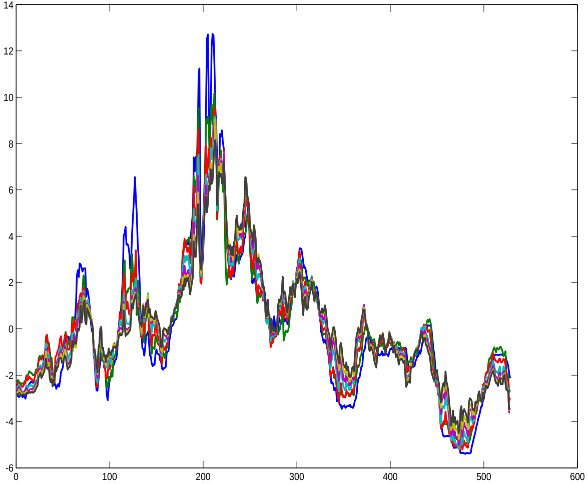

Figure: Bond yield data

Step 1

Normalize variances

$$\widehat{y}_{n,t} = \frac{y_{n,t}}{\sqrt{E y_{n,t} y'_{n,t}}}$$

This is done to ensure that factors structure is not sensitive to unit of measurement

- Example: GDP measured in Euros and Pesetas

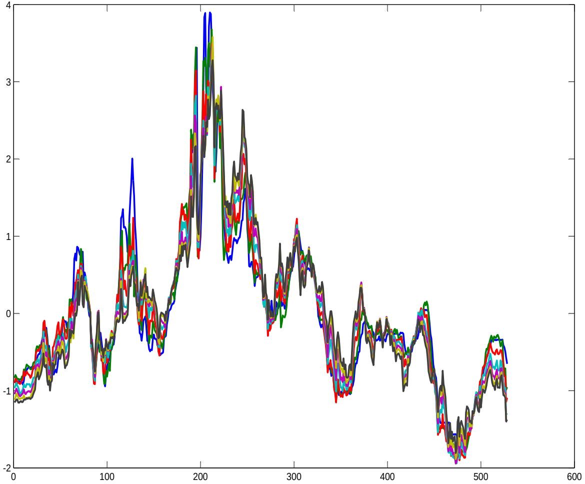

Figure: Normalized Bond yield data

Step 2

$$EY_t Y_t' = \Sigma_{\widehat{y}} = W \Lambda W'$$

Eigenvalue decomposition of covariance matrix in MatLab

$$[W, \Lambda]=\texttt{eig}(\Sigma_{\widehat{y}})$$

gives Λ with eigenvalues in *ascending* order.

- Confusing, since in the literature the *first* eigenvalue usually means the largest eigenvalue
- We will stick to this convention

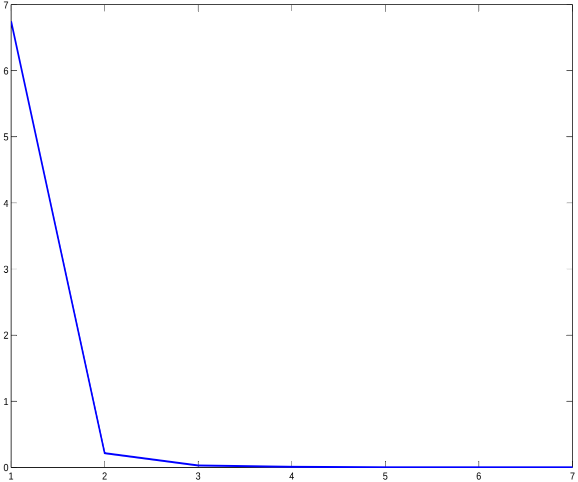

Figure: Scree plot

Step 3

Get the factors

Since

$$\widehat{Y}_t = WF_t$$

and $W^{-1} = W'$ we can get the factors from

$$F_t = W'\widehat{Y}_t$$

The three top rows of F_t contains the principal components (i.e. the factors) associated with the three largest eigenvalues.

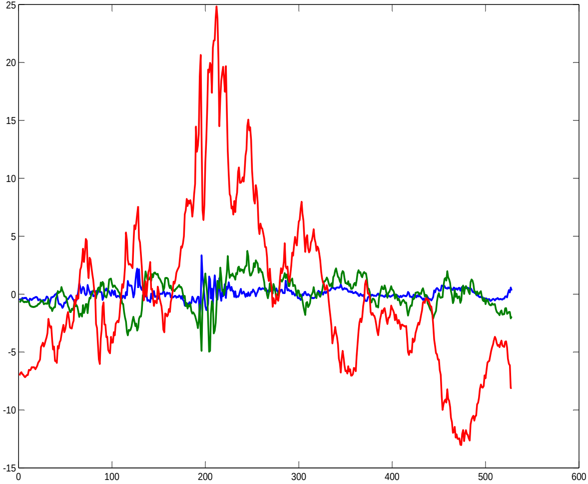

Figure: Time series of (first) three factors

Let's check how we are doing

If three factor model is a good representation of the data, three factors should be able to fit the data well

$$\begin{aligned} \widehat{Y}_t &= WF_t \\ &= \begin{bmatrix} w_1 & \cdots & w_r \end{bmatrix} \begin{bmatrix} f_{1,t} \\ \vdots \\ f_{r,t} \end{bmatrix} \end{aligned}$$

Let's plot the fitted values using only the first factors

$$\widehat{Y}_t^{fit3} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} f_{1,t} \\ f_{2,t} \\ f_{3,t} \end{bmatrix}$$

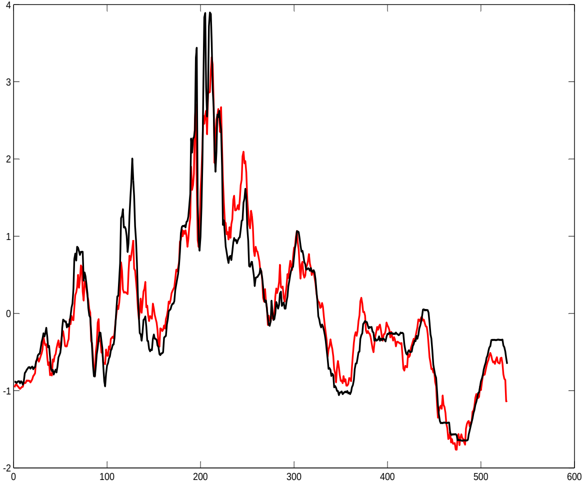

Figure: Fit with 1 factor

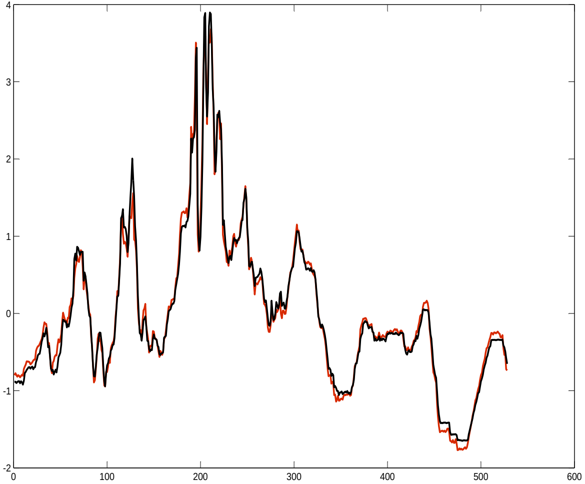

Figure: Fit with 2 factors (Factor 1 + 2)

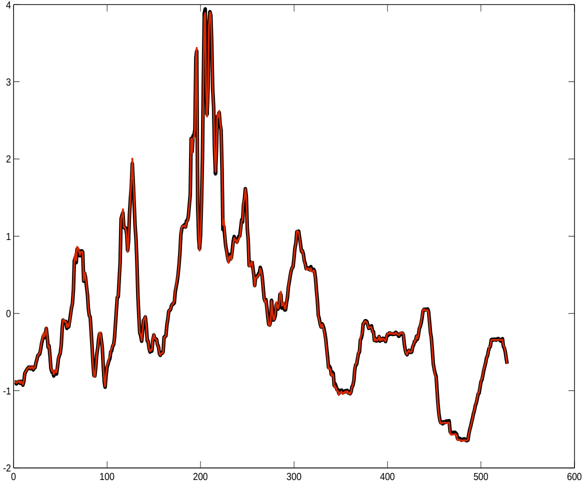

Figure: Fit with 3 factors (Factor 1 + 2 + 3)

Step 4

Estimate the dynamic evolution of the factors

$$
\begin{aligned}
F_{3,t} &= AF_{3,t-1} + u_t \\
F_{3,t} &\equiv \left[\begin{array}{ccc} f_{1,t} & f_{2,t} & f_{3,t} \end{array}\right]'
\end{aligned}
$$

This can be done OLS so that

$$
A = \sum_{t=2}^{T} F_{3,t} F_{3,t-1} \left[\sum_{t=2}^{T} F_{3,t-1} F_{3,t-1} \right]^{-1}
$$

What have we achieved?

We now have an estimated model for the dynamics of $\widehat{Y}_t$

$$\begin{aligned} \widehat{Y}_t &= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} F_{3,t} \\ F_{3,t} &= AF_{3,t-1} + u_t \end{aligned}$$

where we only estimated $15 + 7\text{x}3 = 36$ parameters (W, A and $Eu_t u_t'$)

- Compare this with the $49 + 28 = 77$ parameters of a 7 variable VAR.

Can we interpret the factors?

Not really, but they have been given names that are suggestive for their effect on the yield curve.

- Plot the columns of $\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}$

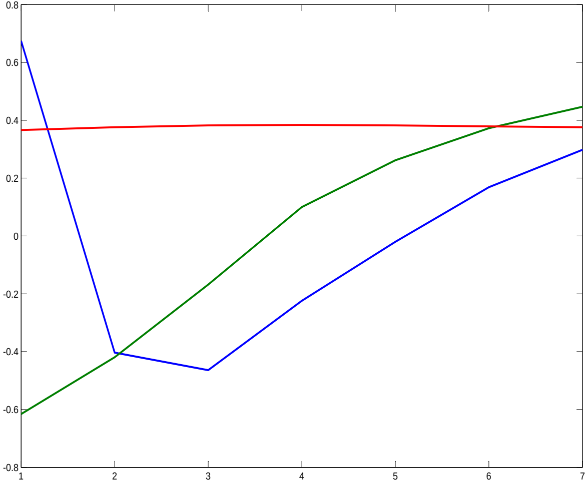

Figure: Factor loadings (Level, slope and curvature)

FAVARS: Combining large data sets with and identifying assumptions

Bernanke, Boivin and Eliasz (QJE 2005) *Measuring the effects of monetary policy: A Factor Augmented Vector Autoregressive (FAVAR) approach*

- Investigate the effects of a monetary policy shock
  - Use more information than what is possible in SVAR
    - Degrees of freedom issues for large N/T
  - Avoid price puzzle
  - IRFs can be computed to more variables

FAVAR implementation

Consider the model

$$\begin{bmatrix} F_t \\ r_t \end{bmatrix} = \Phi \begin{bmatrix} F_{t-1} \\ r_{t-1} \end{bmatrix} + v_t$$

where F_t are some factors and Y_t are some macroeconomic variables of interest.

- The basic premise is that the factors F_t help describe the dynamics of the variables of interest Y_t but also that Y_t have an effect on the factors F_t.

Strategy:

1. Find the factors
2. Estimate reduced form FAVAR
3. Identify the the effects of a policy shock using standard techniques (contemporaneous restrictions)

Step 1: Finding the factors

We want the factors to capture information that is orthogonal to r_t
Start by regressing $\widehat{Y}_t$ on r_t

$$\widehat{Y}_t = \beta r_t + \upsilon_t$$

The factors F_t can then be constructed as the principal components of the residuals υ_t

Step 2: Estimate the FAVAR

Again, this can be done by OLS

$$\begin{bmatrix} F_t \\ r_t \end{bmatrix} = \Phi \begin{bmatrix} F_{t-1} \\ r_{t-1} \end{bmatrix} + v_t$$

Step 3: Identify shock to r_t

With r_t ordered last, this can be done using Cholesky decomposition of errors covariance

$$\widehat{\Omega} = (T-1)^{-1} \sum \left(\begin{bmatrix} F_t \\ r_t \end{bmatrix} - \widehat{\Phi} \begin{bmatrix} F_{t-1} \\ r_{t-1} \end{bmatrix} \right) \times \left(\begin{bmatrix} F_t \\ r_t \end{bmatrix} - \widehat{\Phi} \begin{bmatrix} F_{t-1} \\ r_{t-1} \end{bmatrix} \right)'$$

so that we can recover A_0 and C in

$$A_0 \begin{bmatrix} F_t \\ r_t \end{bmatrix} = A_1 \begin{bmatrix} F_{t-1} \\ r_{t-1} \end{bmatrix} + C\varepsilon_t : \varepsilon_t \sim N(0, I)$$

Step 3: Identify shock to r_t

To compute the impulse response of the variables in Y_t to identified shock, use

$$\frac{\partial \widehat{Y}_{t+s}}{\partial \varepsilon_t^r} = \left[\begin{array}{cc} W & \beta \end{array} \right] \Phi^s C_r$$

since

$$\widehat{Y}_{t+s} - \beta r_{t+s} = W F_t$$

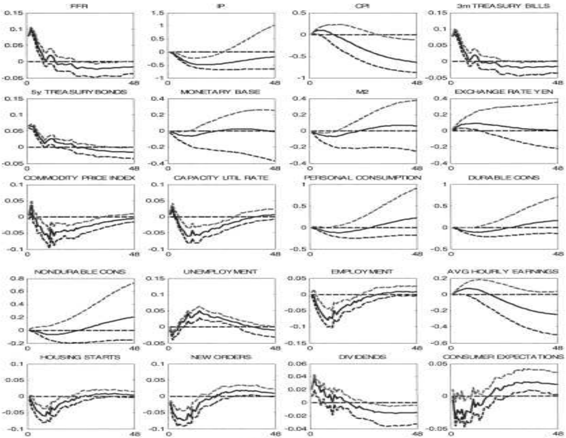

FIGURE II
Impulse Responses Generated from FAVAR with Three Factors and FFR Estimated by Principal Components with Two-Step Bootstrap

That's it for today.