Modelling Information, Learning and Expectations in Macroeconomics

Lecture 10

New York University

Spring 2008

Information, Learning and Expectations in Macroeconomics

Today:

Bounded rationality and adaptive learning -what is it and what are the questions?

Recursive Least Squares Learning

Can we reformulate the RLS Learning as a standard Kalman filter problem?

Next time is Friday 3.30, Room 624

Bounded Rationality

Relax the assumption that agents know the structure of the economy and *"put the agents and the econometrician on the same footing"*

- Can agents discover the rational expectations equilibrium? Bray (JET 1982), Marcet and Sargent (JPE 1989)
- Active learning and monetary policy. Wieland (1998)
- Learning with misspecified models. Sargent (1999)
- Monetary policy rules and indeterminacy. Evans and McGough (JEDC 2005)
- Learning induced business cycle dynamics. Milani (JME 2006), Preston and Eusepi (2007)

What different from the imperfect information set up?

$$
\begin{aligned}
X_t &= A_t X_{t-1} + C_t \mathbf{u}_t \\
Z_t &= D_t X_t + \mathbf{v}_t
\end{aligned}
\tag{1}
$$

Before:

- Coefficient matrices A, C and D assumed known to agents
- State X_t unobservable

Now:

- Agents need to learn about model's coefficients by running least squares regressions (recursively)
- State (or perceived state) assumed to be observable

Problem can still be formulated as an application of the Kalman filter

Language

Two concepts:

- Perceived Law of Motion (PLM)
  - A parameterised equation describing how agents form expectations
- Actual Law of Motion (ALM)
  - An equation describing how the endogenous variables actually evolve
  - The ALM is a function of the PLM
- REE $\Longrightarrow$ PLM=ALM

The Cobweb Model

As simple as it gets:

$$
\begin{aligned}
p_t &= \mu + \alpha E_{t-1} p_t + \eta_t \\
\eta_t &\sim N\left(0, \sigma_\eta^2\right)
\end{aligned}
$$

Can be thought of as a reduced form of the price process for a partial equilibrium model with production delays

$$
\begin{aligned}
REE &: \quad p_t = \frac{\mu}{1-\alpha} + \eta_t \\
PLM &: \quad p_t = m_{t-1} + e_t \\
ALM &: \quad p_t = \mu + \alpha m_{t-1} + \eta_t
\end{aligned}
$$

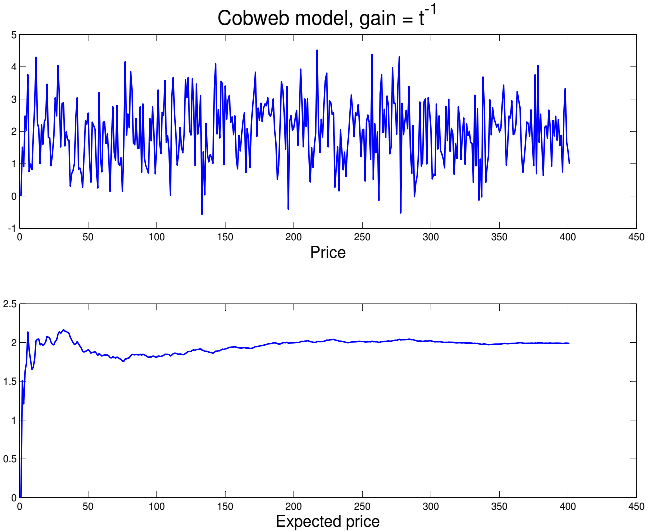
The Cobweb Model

$$\begin{aligned} PLM &: \quad p_t = m_{t-1} + e_t \\ ALM &: \quad p_t = \mu + \alpha m_{t-1} + \eta_t \end{aligned}$$

Agents use least squares to form a belief about m_t

$$\begin{aligned} m_t &= t^{-1} \sum_{s=0}^{t-1} p_{t-s} \\ &= m_{t-1} + t^{-1} \left(p_t - m_{t-1} \right) \end{aligned}$$

$$\begin{bmatrix} p_t \\ m_t \end{bmatrix} = \begin{bmatrix} \mu \\ \mu t^{-1} \end{bmatrix} + \begin{bmatrix} 0 & \alpha \\ 0 & 1 + \alpha t^{-1} - t^{-1} \end{bmatrix} \begin{bmatrix} p_{t-1} \\ m_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ t^{-1} \end{bmatrix} \eta_t$$


Cobweb model, gain = t^-1
5
4
3
2
1
0
-1
0
50
100
150
200
250
300
350
400
450
Price
2.5
2
1.5
1
0.5
0
0
50
100
150
200
250
300
350
400
450
Expected price

Constant gain in the cobweb model

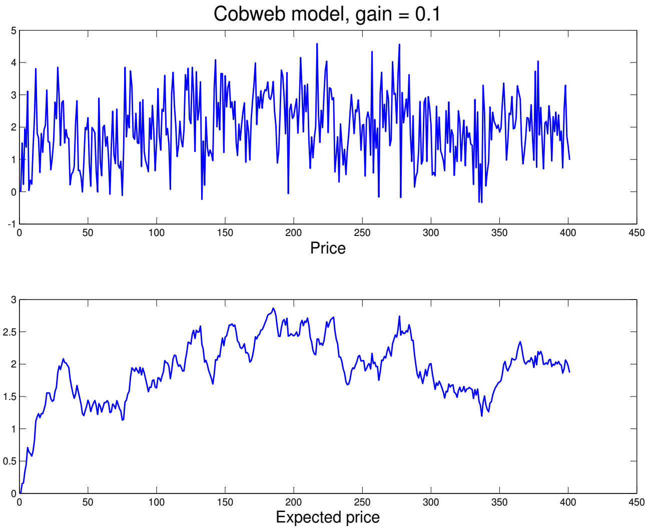
Constant gain $\gamma : 0 < \gamma < 1$. Less weight on old observations can make sense if:

- Agents believe that there is structural change
- Finitely lived agents use t^{-1} gain but are replaced with young agents with a "new" t=1 when they die (Fudenberg and Levine, Econometrica 1993)

$$m_t = m_{t-1} + \gamma \left(p_t - m_{t-1} \right)$$

$$\begin{bmatrix} p_t \\ m_t \end{bmatrix} = \begin{bmatrix} \mu \\ \mu\gamma \end{bmatrix} + \begin{bmatrix} 0 & \alpha \\ 0 & 1 + \alpha\gamma - \gamma \end{bmatrix} \begin{bmatrix} p_{t-1} \\ m_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \eta_t$$

Constant gain give a stationary process for p_t and m_t and is common in empirical (or quasi-empirical) work


Cobweb model, gain = 0.1
Price
Expected price

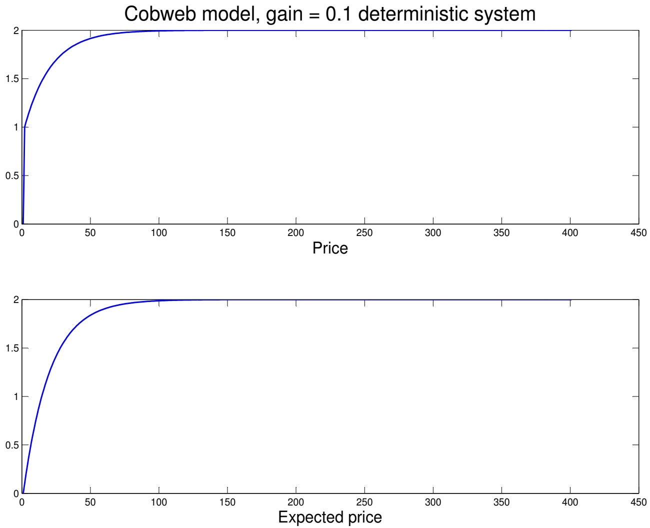
Special case: Constant gain with convergence

Constant gain $\gamma : 0 < \gamma < 1$ and deterministic system:

$$m_t = m_{t-1} + \gamma (p_t - m_{t-1})$$

$$\begin{bmatrix} p_t \\ m_t \end{bmatrix} = \begin{bmatrix} \mu \\ \mu\gamma \end{bmatrix} + \begin{bmatrix} 0 & \alpha \\ 0 & 1 + \alpha\gamma - \gamma \end{bmatrix} \begin{bmatrix} p_{t-1} \\ m_{t-1} \end{bmatrix}$$

The sequence $\{m_t\}_{t=1}^{T}$ tend to the REE solution $\frac{\mu}{1-\alpha}$ as $T \to \infty$



Cobweb model, gain = 0.1 deterministic system
2
1.5
1
0.5
0
0
50
100
150
200
250
300
350
400
450
Price
2
1.5
1
0.5
0
0
50
100
150
200
250
300
350
400
450
Expected price

Cobweb model with observable exogenous variable w_t

$$p_t = \mu + \alpha E_{t-1} p_t + \delta w_{t-1} + \eta_t$$

$$\begin{aligned} REE &: \quad p_t = \frac{\mu}{1-\alpha} + \frac{\delta}{1-\alpha} w_{t-1} + \eta_{t-1} \\ PLM &: \quad p_t = a_{t-1} + b_{t-1} w_{t-1} + e_t \\ ALM &: \quad p_t = \mu + \alpha \left(a_{t-1} + b_{t-1} w_{t-1} \right) + \delta w_{t-1} + \eta_t \end{aligned}$$

Recursive Least Squares Learning

Define vector of parameters in the PLM as

$$\phi_t = \begin{pmatrix} a_t & b_t \end{pmatrix}'$$

and vector of observables as

$$z_t = \begin{pmatrix} 1 & w_t \end{pmatrix}'$$

Normally we would estimate ϕ by OLS

$$\phi_t = \left(\sum_{s=1}^{t} z_s z_s' \right)^{-1} \left(\sum_{s=1}^{t} z_s p_s \right)$$

but we can equivalently estimate ϕ recursively using

$$\begin{aligned} \phi_t &= \phi_{t-1} + t^{-1} R_t^{-1} z_{t-1} \left(p_t - \phi_{t-1}' z_{t-1} \right) \\ R_t &= R_{t-1} + t^{-1} \left(z_{t-1} z_{t-1}' - R_{t-1} \right) \end{aligned}$$

Convergence to REE criteria: E-stability

Stochastic approximation methods can be used to show that if

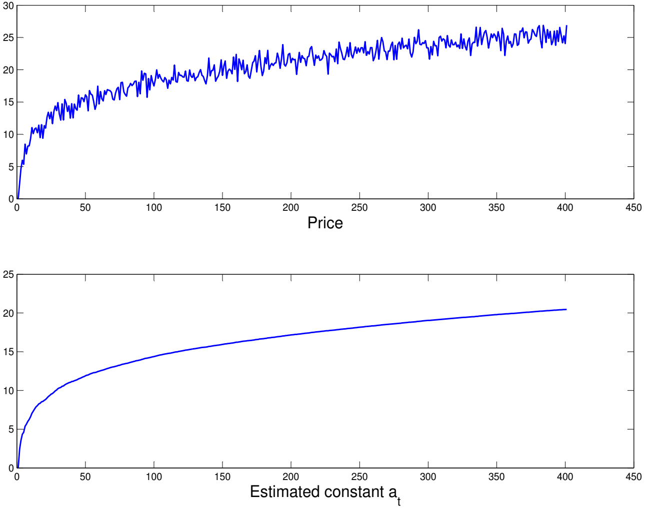
$$\begin{aligned} PLM &: \quad p_t = a_{t-1} + b_{t-1} w_{t-1} + e_t \\ ALM &: \quad p_t = \mu + \alpha \left(a_{t-1} + b_{t-1} w_{t-1} \right) + \delta w_{t-1} + \eta_t \\ &= \quad T(\phi) \end{aligned}$$

$$\begin{aligned} \frac{d\phi}{d\tau} &= T(\phi) - \phi \\ &= \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \alpha - 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \end{aligned}$$

- If $|T(\phi) - \phi| < 0$ $PLM \implies ALM$ with probability 1.
- If $|T(\phi) - \phi| > 0$ $PLM \implies ALM$ with probability 0.

See Evans and Honkapohja (2001).

Cobweb model, $\alpha > 1$



In what sense is learning optimal?

With decreasing gain ($\gamma_t = t^{-1}$) agents do not throw away information and estimates are BLUE

With constant gain ($\gamma_t = \gamma$) agents do not use information efficiently, but can be close to optimal in models where time varying parameters

Perhaps these two points can be understood better by reformulating the RLS algorithm as a Kalman filter problem.

The Kalman filter and Learning

$$\begin{aligned} X_t &= A_t X_{t-1} + C_t \mathbf{u}_t \\ Z_t &= D_t X_t + \mathbf{v}_t \end{aligned}$$

$$X_{t|t} = AX_{t-1|t-1} + K\left(Z_t - AX_{t-1|t-1}\right) \tag{2}$$

(2) looks a bit like

$$\phi_t = \phi_{t-1} + t^{-1} R_t^{-1} z_{t-1}\left(p_t - \phi'_{t-1} z_{t-1}\right)$$

Can we redefine state space to conform to estimating ϕ_t instead of X_t?

RLS as Kalman filtering

New state space system

$$\begin{aligned} \phi_t &= \phi_{t-1} \\ p_t &= z_t' \phi_t + e_t \end{aligned}$$

$$\begin{aligned} \phi_{t|t} &= \phi_{t-1|t-1} + K_t \left(p_t - \phi_{t-1|t-1} \right) \\ K_t &= P_{t|t-1} z_t \left(z_t' P_{t|t-1} z_t + t^{-1} \Sigma \left[e_t e_t \right] \right)^{-1} \\ P_{t|t-1} &= P_{t-1|t-2} - \\ & P_{t-1|t-2} z_t \left(z_t' P_{t-1|t-2} z_t + t^{-1} \Sigma \left[e_t e_t \right] \right)^{-1} z_t' P_{t-1|t-2} \end{aligned}$$

$P_{t|t-1} = P_{t-1|t-1} < P_{t-1|t-2}$ and $\lim_{t \to \infty} P_{t|t-1} = 0 \implies \lim_{t \to \infty} K_t = 0.$

Non-decreasing gain learning as Kalman filtering

Parameters random walk:

$$\begin{aligned}\phi_t &= \phi_{t-1} + \varepsilon_t \\ p_t &= z_t' \phi_t + e_t\end{aligned}$$

$$\begin{aligned}\phi_{t|t} &= \phi_{t-1|t-1} + K_t \left(p_t - \phi_{t-1|t-1}\right) \\ K_t &= P_{t|t-1} z_t \left(z_t' P_{t|t-1} z_t + t^{-1} \Sigma \left[e_t e_t\right]\right)^{-1} \\ P_{t|t-1} &= P_{t-1|t-2} - \\ &\quad P_{t-1|t-2} z_t \left(z_t' P_{t-1|t-2} z + t^{-1} \Sigma \left[e_t e_t\right]\right)^{-1} z_t' P_{t-1|t-2} + \Sigma_{\varepsilon\varepsilon}\end{aligned}$$

K_t will not converge as measurement equation depends on realisations of exogenous process w_t, i.e. $z_t = \begin{pmatrix} 1 & w_t \end{pmatrix}'$

Summing up

Adaptive learning:

- Agents behave as econometricians
- Agents can discover REE if fitting the correct functional form and model is E-stable
- Learning is optimal in the sense of no information wasted if $\gamma_t = t^{-1}$ and actual parameters fixed
- Constant gain not optimal, but makes sense if there are structural breaks or parameter drift
  - Constant gain yields stationary distributions of endogenous variables which is useful for empirical work

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