Modelling Information, Learning and Expectations in Macroeconomics

Lecture 6

New York University

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Private/heterogenous/dispersed information

Today:

- *Forecasting the Forecasts of Others* (Townsend JPE 1983)
- *The Social Value of Public Information* (Morris and Shin AER 2002).

Private/heterogenous/dispersed information

Why is this interesting?

- Seems realistic: Allows for disagreement between agents in model

Is it important?

- We need to solve models to answer that question

The Infinite Regress of Expectations

Think of an agent j that cares about some unobservable state variable /*theta* but also about the average action taken by other agents so that

$$\max_{a(j)} U(j) = -r\left(a(j) - \theta\right)^2 - (1 - r)\left(a(j) - \int a(j)dj\right)^2$$

This is (almost) the model of Morris and Shin (2002). But let's be chronological and start with Townsend (1983)

Forecasting the Forecasts of Others, Townsend JPE 1983

Output sector i

$$y_t^i = f_0 k_t^i$$

Market clearing price

$$\begin{aligned} P_t^i &= -b_1 Y_t^i + z_t^i \\ z_t &= \theta_t + \epsilon_t^i \end{aligned}$$

where

$$\theta_t = \rho\theta_{t-1} + v_t$$

and $\epsilon_t^i \sim N(0, \sigma_\epsilon^2)$ and $v_t \sim N(0, \sigma_v^2)$

Forecasting the Forecasts of Others, Townsend JPE 1983

The firms profit max problem:

$$\max_{\{k_t^i\}_{t=1}^{\infty}} E_0^i \sum_{t=0}^{\infty} \beta^t \left[P_t^i f_0 k_t^i - \frac{f_1}{2} \left(k_t^i\right)^2 - \frac{f_2}{2} \left(k_{t+1}^i - k_t^i\right)^2 \right]$$

$$f_0, f_2 \quad > \quad 0, \qquad f_1 \geq 0$$

Decision rule

$$k_{t+1} = \lambda_1 k_t^i + \frac{f_0 \beta \lambda_1}{f_2} \sum_{j=0}^{\infty} (\beta \lambda_1)^j E\left(P_{t+1+j}^i \mid \Omega_t^i\right)$$

gives a law of motion for aggregate industry i capital stock

$$K_{t+1}^i = h_1 K_t^1 + h_2 M_t^i$$

where

$$M_t^i = E\left(\theta_t \mid \Omega_t^i\right)$$

Hierarchical information structure

- Industry 1 is "self contained":
  - Does not observe prices in any other industry
- Industry 2 observes prices in Industry 1, but there is no trade or other "real" interaction across sectors
  - Industry 2 uses observation of price in Industry 1 to form an estimate of θ_t

The only link between industries are that they both try to estimate the same unobservable state

Industry 1 filtering problem

A firm in Industry 1 observes

$$
\begin{aligned}
z_0^1 &= \theta_0 + \epsilon_0^1 \\
z_1^1 &= \rho\theta_0 + v_1 + \epsilon_1^1 \\
&\vdots \\
z_t^1 &= \rho^t\theta_0 + \rho^{t-1}v_1 + \ldots + v_t + \epsilon_t^1
\end{aligned}
$$

We know how to estimate this, right?

Industry 1 filtering problem

By brute force:

$$
\begin{aligned}
& E\left(\begin{bmatrix} \theta_0 & v_1 & \cdots & v_t \end{bmatrix}' \mid z_t^1, z_{t-1}^1, ..., z_0^1 \right) \\
= \; & E\left(\begin{bmatrix} \theta_0 & v_1 & \cdots & v_t \end{bmatrix}' \begin{bmatrix} z_0^1 & z_1^1 & \cdots & z_t^1 \end{bmatrix} \right) \times \\
& \left[E\left(\begin{bmatrix} z_0^1 & z_1^1 & \cdots & z_t^1 \end{bmatrix}' \begin{bmatrix} z_0^1 & z_1^1 & \cdots & z_t^1 \end{bmatrix} \right) \right]^{-1}
\end{aligned}
$$

so that

$$
\begin{aligned}
M_t^1 &= W_t E\left(\begin{bmatrix} \theta_0 & v_1 & \cdots & v_t \end{bmatrix}' \mid z_t^1, z_{t-1}^1, ..., z_0^1 \right) \\
W_t &= \begin{bmatrix} \rho^t & \rho^{t-1} & \cdots & 1 \end{bmatrix}
\end{aligned}
$$

Industry 1 filtering problem

By Kalman filtering

$$M_t^1 = \rho M_{t-1}^1 + G_t \left(z_t^1 - \rho M_{t-1}^1 \right)$$

or with Townsend's notation:

$$M_t^1 = \alpha_0 M_{t-1}^1 + \alpha_1 z_t^1$$

Industry 1 law of motion

$$\begin{bmatrix} K^1_{t+1} \\ M^1_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & 0 \\ 0 & \alpha_0 & \alpha_1 \rho \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_1 v_{t+1} + \alpha_1 \epsilon^1_{t+1} \\ v_{t+1} \end{bmatrix}$$

Given the decision rule, industry 1 is solved.

Industry 2 filtering problem

Firms in Industry 2 also want to estimate the current value of θ_t and they observe

$$
\begin{aligned}
z_0^2 &= \theta_0 + \epsilon_0^2 \\
z_1^2 &= \rho\theta_0 + v_1 + \epsilon_1^2 \\
&\vdots \\
z_t^2 &= \rho^t\theta_0 + \rho^{t-1}v_1 + ... + v_t + \epsilon_t^2
\end{aligned}
$$

as well as the history of prices in industry 1 so Industry 2's information set is

$$\Omega_t^2 = \left\{z_s^2, P_s^1, P_s^2, M_s^2, K_s^2 : s = 0, 1, 2, ..., t\right\}$$

Industry 2 filtering problem

$$\begin{bmatrix} K^1_{t+1} \\ M^1_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & 0 \\ 0 & \alpha_0 & \alpha_1 \rho \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_1 v_{t+1} + \alpha_1 \epsilon^1_{t+1} \\ v_{t+1} \end{bmatrix}$$

$$\begin{bmatrix} z^2_t \\ P^1_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -b_1 f_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} \epsilon^2_t \\ \epsilon^1_t \end{bmatrix}$$

This is standard form Kalman filter.

Industry 2 filtering problem

We can then write the complete system as

$$\begin{bmatrix} K^1_{t+1} \\ M^1_{t+1} \\ \theta_{t+1} \\ K^2_{t+1} \\ M^2_{t+1} \\ E\left[K^1_{t+1} \mid \Omega^2_{t+1}\right] \\ E\left[M^1_{t+1} \mid \Omega^2_{t+1}\right] \end{bmatrix} = A \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \\ K^2_t \\ M^2_t \\ E\left[K^1_t \mid \Omega^2_t\right] \\ E\left[M^1_t \mid \Omega^2_t\right] \end{bmatrix} + C \begin{bmatrix} v_t \\ \epsilon^1_t \\ \epsilon^2_t \end{bmatrix}$$

What if firms in industry 1 also can observe prices in Industry 2?

Then we need to include industry 1's expectations about industry 2 expectations about .....industry 1's expectations of θ_t

- This is the infinite regress of expectations.

Townsend suggested that we

- Assume
  - that shocks are observed perfectly with a finite lag and
  - that there is a continuum of industries.
- Use projection methods
  - This works since projections on a finite dimensional observations vector now spans the state

Outline of Townsend's method:

A firm in Industry i observes

$$\Omega_t^i = \left\{ z_s^i, \overline{P}_s, \theta_{s-2} : s = 0, 1, 2, ..., t; \right\}$$

We then have

$$\begin{aligned} M_t^i &= E\left[\theta_t \mid \Omega_t^i\right] \\ &= E\left(\theta_t \left[\begin{array}{ccccc} z_t^i & z_{t-1}^i & \overline{P}_t & \overline{P}_{t-1} & \theta_{t-2} \end{array}\right]\right) \\ &\quad \times \Bigg(\left[\begin{array}{ccccc} z_t^i & z_{t-1}^i & \overline{P}_t & \overline{P}_{t-1} & \theta_{t-2} \end{array}\right]' \left[\begin{array}{ccccc} z_t^i & z_{t-1}^i & \overline{P}_t & \overline{P}_{t-1} & \theta_{t-} \end{array}\right. \end{aligned}$$

and as before the $K_{t+1}^i = h_1 K_t^1 + h_2 M_t^i$. However, we do not have an analytical expression for the covariances above, so model need to be solved as a fixed point problem. (Townsend used direct function iteration.)

What about the law of iterated expectations and higher order expectations?

The law of iterated expectations does not apply to higher order expectations

- It can be rational to think that others are wrong

Consider

$$\begin{aligned} \theta_t &= v_t \\ z_t^i &= v_t + \epsilon_t^i \end{aligned}$$

then

$$\begin{aligned} E\left[\theta_t \mid z_t^i\right] &= \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\epsilon^2} z_t^i \\ &= g z_t^i \end{aligned}$$

What about the law of iterated expectations and higher order expectations?

Define the average expectation as Y_t

$$\begin{aligned} Y_t &= \int E\left[\theta_t \mid z_t^i\right] di \\ &= g\theta_t + g \int \epsilon_t^i di \\ &= g\theta_t \end{aligned}$$

What about the law of iterated expectations and higher order expectations?

We can treat Y_t as any other random variable and let agents form an estimate of the average estimate

$$
\begin{aligned}
E\left[Y_t \mid z_t^i\right] &= g \times E\left[\theta_t \mid z_t^i\right] \\
&= g \times g z_t^i \\
&= E\left[\int E\left[\theta_t \mid z_t^i\right] di \mid z_t^i\right]
\end{aligned}
$$

so that

$$
\int E\left[\int E\left[\theta_t \mid z_t^i\right] di \mid z_t^i\right] di = g^2 \theta_t
$$

and so on so that an average k order expectation of θ_t is given by $g^k \theta_t$

Morris and Shin's model

$$
\begin{aligned}
\max_{a_i} U_i &= -r\left(a_i-\theta\right)^2-(1-r)\left(L_i-\overline{L}\right)^2 \\
0 &< r<1 \\
L_i &\equiv \int\left(a_j-a_i\right)^2 dj \\
\overline{L} &\equiv \int L_j dj
\end{aligned}
$$

Two signals of θ

Public signal y

$$
\begin{aligned}
y &= \theta + \eta \\
\eta &\sim N(0, \sigma_\eta^2)
\end{aligned}
$$

private signal x_i

$$
\begin{aligned}
x_i &= \theta + \varepsilon_i \\
\varepsilon_i &\sim N(0, \sigma_\varepsilon^2) \forall i \\
E\varepsilon_i\varepsilon_j &= 0 : i \neq j
\end{aligned}
$$

Player i's best response

$$a_i = (1-r)E_i(\theta) + rE_i(\bar{a})$$

repeated substitution yields

$$a_i = (1-r)\sum_{k=1}^{\infty} r^k \theta^{(k)}(i)$$

where

$$\begin{aligned}
\theta^{(1)}(i) &\equiv E\left[\theta \mid I(i)\right] \\
\theta^{(2)}(i) &\equiv E\left[\int \theta^{(1)}(i)di \mid I(i)\right] \\
\theta^{(k)}(i) &\equiv E\left[\int \theta^{(k-1)}(i)di \mid I(i)\right]
\end{aligned}$$

Solving for average actions

Take averages of actions across agents

$$\bar{a} = (1-r)\sum_{k=1}^{\infty} r^k \theta^{(k)}$$

We can normalise by the public signal so that we get back to set up above. Define (re-normalised) private signal as

$$\widehat{x}_i = x_i - y$$

then

$$E\left[\theta_t \mid \widehat{x}_i\right] = g\left(x_i - y\right) + y$$

where $g = \sigma_\eta^2/(\sigma_\eta^2 + \sigma_\varepsilon^2)$. Higher order expectations of θ are then given by

$$\theta^{(k)} = g^k\left(x_i - y\right) + y$$

Solving for average actions

Substitute

$$\theta^{(k)} = g^k (\theta - y) + y$$

into

$$\bar{a} = (1 - r) \sum_{k=1}^{\infty} r^k \theta^{(k)}$$

to get

$$\begin{aligned} \bar{a} - y &= -(1 - r) \sum_{k=1}^{\infty} r^k g^k \eta \\ \bar{a} - \theta &= \left(1 - \frac{(1 - r)}{(1 - rg)} \right) \eta \end{aligned}$$

Figure: Impact of public signal noise