Modelling Information, Learning and Expectations in Macroeconomics

Lecture 7

New York University

Spring 2008

Private/heterogenous/dispersed information

- Private information, strategic interaction
  - Every agent has his own "window to the world" but no agent is better informed than others
  - Individual pay offs depend on (average) action taken by others
- The principal modeling difficulty: The infinite regress of "forecasting the forecasts of others" (Townsend 1983)

Solving dynamic models with private information

Last week:

- Short lived private info (Townsend 1983)
  - Not always realistic to assume that shocks can be observed with a lag
- Static model (Morris and Shin 2002)

Today:

- Static choices, dynamic filtering (Woodford 2002)
- Dynamic choices, dynamic filtering (Nimark 2007).

Woodford's model of nominal price adjustment

Motivation:

- Why do real variables respond sluggishly to nominal shocks?

A simple framework with strong assumptions:

- No endogenous variables are observed
- No dynamic choices, i.e. no Euler-type equations
  - Output (and consumption) determined by assuming exogenous nominal GDP and constant velocity-of-money

Woodford's model of nominal price adjustment

A variant of Lucas-Island model

$$y_t = \alpha \left(q_t - q_{t|t} \right)$$

Optimal firm i price given by

$$p_t(i) = p_{t|t}(i) + \xi y_{t|t}(i)$$

where

$$p_{t|t}(i) = E \left[p_t \mid I_t(i) \right]$$

and $0 < \xi < 1$.
Intuition: Firm i marginal cost depend on relative price and aggregate output (CES type utility)

Notation

Agents are indexed by $i \in (0,1)$ and q_t is nominal GDP

$$
\begin{aligned}
q_{t|t}^{(0)} &\equiv q_t \\
q_{t|t}^{(1)} &\equiv \int E\left[q_{t|t}^{(0)} \mid I_t(i)\right] \, di \\
q_{t|t}^{(2)} &\equiv \int E\left[q_{t|t}^{(1)} \mid I_t(i)\right] \, di \\
q_{t|t}^{(k)} &\equiv \int E\left[q_{t|t}^{(k-1)} \mid I_t(i)\right] \, di
\end{aligned}
$$

Woodford's model of nominal price adjustment

Take averages of optimal price across agents and use that $y_t = q_t - p_t$ to get

$$\begin{aligned} \int p_t(i) &= \int p_{t|t}(i)di + \int \xi y_{t|t}(i)di \\ &= \xi q_{t|t} + (1-\xi)p_{t|t} \end{aligned}$$

repeated substitution of higher order expectations of the price level into the actual price level expression yields

$$p_t = \sum_{k=1}^{\infty} \xi \left(1-\xi\right)^{k-1} q_{t|t}^{(k)}$$

again using $y_t = q_t - p_t$

$$y_t = \sum_{k=1}^{\infty} \xi \left(1-\xi\right)^{k-1} \left[q_t - q_{t|t}^{(k)} \right]$$

The Process for Nominal GDP

$$\Delta q_t = (1-\rho)\, g + \rho \Delta q_{t-1} + u_t$$

or

$$\begin{aligned} X_t &\equiv \begin{bmatrix} q_t \\ q_{t-1} \end{bmatrix} \\ X_t &= c + AX_{t-1} + au_t \end{aligned}$$

where

$$c \equiv \begin{bmatrix} (1-\rho)\, g \\ 0 \end{bmatrix}, A \equiv \begin{bmatrix} 1+\rho & -\rho \\ 1 & 0 \end{bmatrix}, a \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Under full information X_t completely characterizes the state.

The state with private information

Agent i observes the signal $z_t(i)$

$$z_t(i) = q_t + v_t(i)$$

Woodford has a nice trick: Define a new state vector $\overline{X}_t$

$$\overline{X}_t = \begin{bmatrix} X_t \\ F_t \end{bmatrix}$$

where

$$F_t \equiv \sum_{k=1}^{\infty} \xi (1 - \xi)^{k-1} X_t^{(k)}$$

The law of motion (LOM) of the state with private information

Conjecture a law of motion for $\overline{X}_t$

$$\overline{X}_t = \overline{c} + M\overline{X}_{t-1} + mu_t$$

or

$$\begin{bmatrix} X_t \\ F_t \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} A & 0 \\ G & H \end{bmatrix} \begin{bmatrix} X_{t-1} \\ F_{t-1} \end{bmatrix} + \begin{bmatrix} a \\ h \end{bmatrix} u_t$$

We thus need to find d, G, H and h to solve the model as

$$\begin{aligned} p_t &= \sum_{k=1}^{\infty} \xi \left(1-\xi\right)^{k-1} q_{t|t}^{(k)} \\ &= e_3' \overline{X}_t \\ &= e_1' F_t \end{aligned}$$

The LOM of the state with private information

Start by re-writing the signal $z_t(i)$ as

$$z_t(i) = e_1 \overline{X}_t + v_t(i)$$

we then have an updating equation for agent i's estimate of $\overline{X}_t$

$$\begin{aligned} \overline{X}_{t|t} &= M\overline{X}_{t-1|t-1} + ke_1' \left(\overline{X}_t - M\overline{X}_{t-1|t-1} \right) \\ &= \overline{c} + ke_1' M\overline{X}_{t-1} + \left(I - ke_1' \right) M\overline{X}_{t-1|t-1} + ke_1' mu_t \end{aligned}$$

Use that

$$\begin{aligned} F_t &= \overline{\xi X}_{t|t} \\ \overline{\xi} &\equiv \begin{bmatrix} \xi \mathbf{I}_2 & (1-\xi)\, \mathbf{I}_2 \end{bmatrix} \end{aligned}$$

to get

$$F_t = \overline{\xi}\overline{c} + \overline{\xi} ke_1' M\overline{X}_{t-1} + \overline{\xi} \left(I - ke_1' \right) M\overline{X}_{t-1|t-1} + \overline{\xi} ke_1' mu_t$$

The LOM of the state with private information

Substitute in the partitioned law of motion for M in

$$F_t = \overline{\xi}\overline{c} + \overline{\xi} k e_1' M \overline{X}_{t-1} + \overline{\xi}\left(I - k e_1'\right) M \overline{X}_{t-1|t-1} + \overline{\xi} k e_1' m u_t$$

to get

$$\begin{aligned} F_t &= \xi c + (1-\xi) d + \xi k A_1 X_{t-1} + \\ & [\xi A + (1-\xi) G - \xi k A_1] X_{t-1|t-1} + \\ & (1-\xi) H F_{t-1|t-1} + \xi k u_t \end{aligned}$$

We want to get rid of $F_{t-1|t-1}$

The LOM of the state with private information

To get rid of $F_{t-1|t-1}$ use

$$\begin{aligned} F_t &= \left[\begin{array}{cc} \xi \mathbf{I}_2 & (1-\xi)\,\mathbf{I}_2 \end{array}\right]\left[\begin{array}{c} X_{t|t} \\ F_{t|t} \end{array}\right] \\ &\Longrightarrow \quad (1-\xi)\,F_{t-1|t-1} = F_{t-1} - \xi X_{t-1|t-1} \end{aligned}$$

Substitute last line into equation for F_t

$$\begin{aligned} F_t &= \xi c + (1-\xi)d + \xi k A_1 X_{t-1} + \\ &\quad [\xi A + (1-\xi)G - \xi H - \xi k A_1]\,X_{t-1|t-1} + \\ &\quad H F_t + \xi k u_t \end{aligned}$$

The LOM of the state with private information

$$
\begin{aligned}
F_t &= \xi c + (1-\xi)d + \xi k A_1 X_{t-1} + \\
&\quad [\xi A + (1-\xi)G - \xi H - \xi k A_1]\, X_{t-1|t-1} + \\
&\quad H F_t + \xi k u_t
\end{aligned}
$$

Equate coefficients in lower block of

$$
\begin{bmatrix} X_t \\ F_t \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} A & 0 \\ G & H \end{bmatrix} \begin{bmatrix} X_{t-1} \\ F_{t-1} \end{bmatrix} + \begin{bmatrix} a \\ h \end{bmatrix} u_t
$$

to get $d = \xi c + (1-\xi)d, \quad G = \xi k A_1, h = \xi k$ and $H = A - \xi k A_1$.

The LOM of the state with private information

Finding the Kalman gain k still remains. It is given by the usual Kalman filter formulas for the system

$$\begin{aligned} \begin{bmatrix} X_t \\ F_t \end{bmatrix} &= \bar{c} + M \begin{bmatrix} X_{t-1} \\ F_{t-1} \end{bmatrix} + m u_t \\ z_t(i) &= e_1' \begin{bmatrix} X_t \\ F_t \end{bmatrix} + v_t(i) \end{aligned}$$

so that

$$\begin{aligned} k &= p e_1' \left(e_1' p e_1 + \sigma_v^2 \right)^{-1} \\ p &= M \left(p - p e_1' \left(e_1' p e_1 + \sigma_v^2 \right)^{-1} e_1 p \right) M' + m \sigma_u^2 m' \end{aligned}$$

Solve for M, m, k and p by direct iteration.

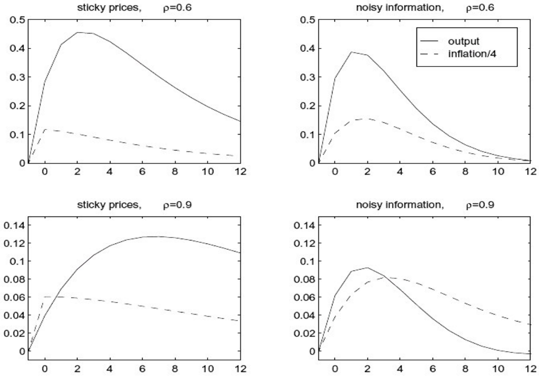


Figure 5: The comparison extended to the cases $\rho = .6$ and $.9$.

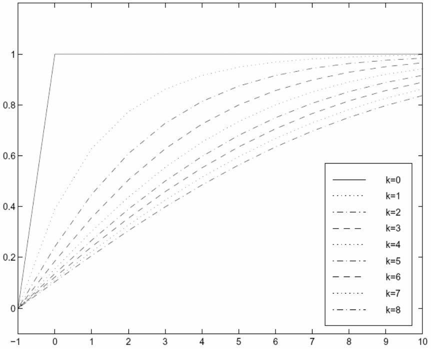


Figure 3: Impulse response functions for higher-order expectations $q_t^{(k)}$, for various values of k. The case $k = 0$ indicates the exogenous disturbance to log nominal GDP itself.

Summing up Woodford's solution method

A clever way of reducing state dimension by recognizing that a particular weighted sum of higher order expectations are sufficient to determine the price level

- Static choices (no Euler-type equations)
  - All prices adjusted in each period
  - Output/consumption determined by after prices are set by exogenous process for nominal GDP
- The filtering problem is simplified by assuming that no endogenous observable variables
  - Method can be amended to include endogenous signals, but more shocks are then needed

Solving models with private information and dynamic choices

The strategy of Nimark (2007)

- Impose structure on higher order expectations by assuming it is common knowledge that agents are rational Bayesians
  - By it self does not solve the "infinite regress problem" but makes thinking about higher order expectations tractable
- Show that impact of expectations diminishes as order increases which allows for an arbitrarily good approximation
  - Use Singleton's (1987) model of asset pricing with disparately informed traders as vehicle for the argument

Notation

Agents are indexed by $j \in (0,1)$

$$
\begin{aligned}
\theta_{t|t}^{(0)} &\equiv \theta_t \\
\theta_{t|t}^{(1)} &\equiv \int E\left[\theta_{t|t}^{(0)} \mid I_t(j)\right] \quad dj \\
\theta_{t|t}^{(2)} &\equiv \int E\left[\theta_{t|t}^{(1)} \mid I_t(j)\right] \quad dj \\
\theta_{t|t}^{(k)} &\equiv \int E\left[\theta_{t|t}^{(k-1)} \mid I_t(j)\right] \quad dj
\end{aligned}
$$

Notation cont.

Denote a vector consisting of a *hierarchy of expectations* (from order zero to k)

$$\theta_{t|t}^{(0:k)} = \begin{bmatrix} \theta_{t|t}^{(0)} \\ \theta_{t|t}^{(1)} \\ \vdots \\ \theta_{t|t}^{(k)} \end{bmatrix}$$

Constructing a law of motion for expectations using common knowledge of rational expectations

- Illustrate how common knowledge of (Bayesian) rationality impose structure on higher order expectations
- A simple example (no economics yet)

Estimating an unobservable process

$$\theta_t = \rho\theta_{t-1} + v_t$$

In period t agent j observes the private noisy signal $s_t(j)$

$$\begin{aligned} s_t(j) &= \theta_t + \eta_t(j), \\ \eta_t(j) &\sim N\left(0, \sigma_\eta^2\right) \ \forall \ j \end{aligned}$$

Updating equation

$$\theta_{t|t}^{(1)}(j) = (1 - g_1)\,\rho\theta_{t-1|t-1}^{(1)}(j) + g_1 s_t(j)$$

Higher order estimates

A new state space system

$$\begin{bmatrix} \theta_t \\ \theta^{(1)}_{t|t} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ g_1\rho & (1-g_1)\rho \end{bmatrix} \begin{bmatrix} \theta_{t-1} \\ \theta^{(1)}_{t-1|t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ g_1 \end{bmatrix} v_t$$

$$s_t(j) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_t \\ \theta^{(1)}_{t|t} \end{bmatrix} + \eta_t(j)$$

Higher order estimates

A new updating equation

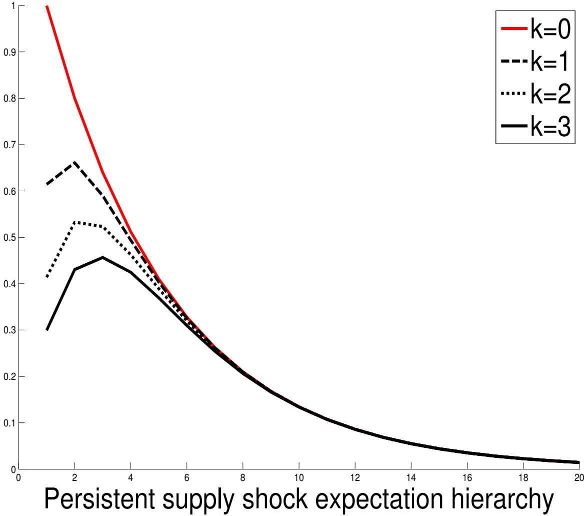
$$\begin{bmatrix} \theta^{(1)}_{t|t}(j) \\ \theta^{(2)}_{t|t}(j) \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ g_1\rho & (1-g_1)\rho \end{bmatrix} \begin{bmatrix} \theta^{(1)}_{t-1|t-1}(j) \\ \theta^{(2)}_{t-1|t-1}(j) \end{bmatrix}$$

$$+ \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}' \left(\begin{bmatrix} \theta_t \\ \theta^{(1)}_{t|t} \end{bmatrix} - \begin{bmatrix} \theta^{(1)}_{t-1|t-1}(j) \\ \theta^{(2)}_{t-1|t-1}(j) \end{bmatrix} \right) + \eta_t(j) \right)$$

Higher order estimates

Law of motion for $\theta_{t|t}^{(0:2)}$

$$\begin{bmatrix} \theta_{t|t}^{(0)} \\ \theta_{t|t}^{(1)} \\ \theta_{t|t}^{(2)} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ g_1\rho & (1-g_1)\,\rho & 0 \\ g_2\rho & (g_1-g_2)\rho & (1-g_1)\,\rho \end{bmatrix} \begin{bmatrix} \theta_{t-1|t-1}^{(0)} \\ \theta_{t-1|t-1}^{(1)} \\ \theta_{t-1|t-1}^{(2)} \end{bmatrix} + \begin{bmatrix} 1 \\ g_1 \\ g_2 \end{bmatrix} v_t$$


k=0
k=1
k=2
k=3
Persistent supply shock expectation hierarchy

Bounded variance of higher order expectations

$$\theta_{t|t}^{(k)} \equiv \theta_{t|t}^{(k+1)} + e_t^{(k+1)}$$

$$var\left(\theta_{t|t}^{(k)}\right) = var\left(\theta_{t|t}^{(k+1)}\right) + var\left(e_t^{(k+1)}\right) + 2covar\left(\theta_{t|t}^{(k+1)}, e_t^{(k+1)}\right)$$

If $covar \neq 0$ then a better estimate $\widehat{\theta}_{t|t}^{(k+1)}$ can be found

$$\widehat{\theta}_{t|t}^{(k+1)} = \theta_{t|t}^{*(k+1)} - \frac{covar\left(\theta_{t|t}^{*(k+1)}, e_t^{(k+1)}\right)}{var\left(\theta_{t|t}^{*(k+1)}\right)}\theta_{t|t}^{*(k+1)}$$

Common knowledge of rational expectations thus implies that

$$var\left(\theta_{t|t}^{(k)}\right) \geq var\left(\theta_{t|t}^{(k+1)}\right)$$

The Singleton (1987) asset pricing model

Trader j's demand

$$z_t^d(j) = \frac{(E[p_{t+1} \mid I_t(j)] - (1+\bar{r})\, p_t) + (\bar{c} + \psi c_t)}{\gamma\delta}$$

Supply

$$\begin{aligned} z_t^s &= \xi p_t + \theta_t + \epsilon_t \\ \theta_t &= \rho\theta_{t-1} + v_t \end{aligned}$$

The Singleton asset pricing model

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid I_t(j)\right] \quad dj \right) + \lambda \psi c_t - \delta \gamma \lambda \left[\theta_t + \epsilon_t\right]$$

where

$$\lambda \equiv \frac{1}{\xi \gamma \delta + (1 + \bar{r})}$$

Trader j's information set

$$\begin{aligned} I_t(j) &= \{s_{t-T}(j), p_{t-T}, c_{t-T} : T \geq 0\} \\ s_t(j) &= \theta_t + \eta_t(j) \end{aligned}$$

Singleton:

$$I_t^S = \{s_{t-T}(j), p_{t-T}, c_{t-T} : T \geq 0; v_{t-T}, \epsilon_{t-T} : T \geq 2\}$$

The Full Information Equilibrium Price

Iterate price Euler-equation forward

$$
\begin{aligned}
p_t &= \lambda\psi c_t - \delta\gamma\lambda\left(\theta_t + \epsilon_t\right) + \\
&\quad \left(\lambda\psi\right)^2 c_t - (\delta\gamma\lambda)\lambda\rho\theta_t + \ldots \\
&\quad \ldots + \left(\lambda\psi\right)^\infty c_t - \left(\delta\gamma\lambda\right)\left(\lambda\rho\right)^\infty \theta_{t+\infty}
\end{aligned}
$$

to get

$$
p_t = \frac{\lambda\psi}{1-\lambda\psi} c_t - \frac{\delta\gamma\lambda}{1-\lambda\rho}\theta_t - \delta\gamma\lambda\epsilon_t
$$

The Private Information Equilibrium Price: Complications

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid I_t(j)\right] \quad dj \right) + \lambda\psi c_t - \delta\gamma\lambda\left[\theta_t + \epsilon_t\right]$$

Iterate price Euler-equation forward

$$\begin{aligned} p_t \quad = \quad & \lambda\psi c_t - \delta\gamma\lambda\left(\theta_t + \epsilon_t\right) \\ & + \left(\lambda\psi\right)^2 c_t - (\delta\gamma\lambda)\lambda \int E\left[\theta_{t+1} \mid I_t(j)\right] dj \\ & +(\delta\gamma\lambda)\lambda^2 \int E\left[\int E\left[p_{t+2} \mid I_{t+1}(j)\right] dj \mid I_t(j) \right] dj] \end{aligned}$$

The law of iterated expectations does not apply: Individual traders can rationally believe that average expectations are "incorrect"

An average one-period-ahead expectations operator

Conjecture a law of motion for hierarchy

$$\theta_{t|t}^{(0:\infty)} = M\theta_{t-1|t-1}^{(0:\infty)} + N \begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix}$$

Define new operator $\overline{M} : \mathbb{R}^\infty \to \mathbb{R}^\infty$

$$\overline{M} \equiv M \begin{bmatrix} \mathbf{0}_{\infty\times 1} & I \end{bmatrix}$$

then higher order expectations of future fundamental are given by

$$\int E\left[\theta_{t+1} \mid I_t(j)\right] dj = e_1'\overline{M}\theta_{t|t}^{(0:\infty)}$$

$$\int E\left[\int E\left[\theta_{t+2} \mid I_{t+1}(j)\right] dj \mid I_t(j)\right] dj] = e_1'\overline{M}^2\theta_{t|t}^{(0:\infty)}$$

...and so on.

The price function

The price of the asset can be written as a function of the state

$$p_t = \begin{bmatrix} a_0 & a_1 & \cdots & a_\infty \end{bmatrix} \begin{bmatrix} \theta_{t|t}^{(0)} \\ \theta_{t|t}^{(1)} \\ \vdots \\ \theta_{t|t}^{(\infty)} \end{bmatrix} + \frac{\lambda\psi}{1-\lambda\psi} c_t - \delta\gamma\lambda\epsilon_t$$

where the vector **a** resembles a discounted geometric sum of expected future fundamentals

$$\begin{aligned} \mathbf{a} &= -\delta\gamma\lambda e_1' \left[I - \lambda\overline{M} \right]^{-1} \\ \mathbf{a} &= \begin{bmatrix} a_0 & a_1 & \cdots & a_\infty \end{bmatrix} \end{aligned}$$

A finite state representation

The matrix M in the law of motion of the hierarchy is lower triangular with each row summing to ρ

$$0 \leq |\rho\lambda| < 1 \implies \lim_{n\to\infty} a_n = 0 \tag{1}$$

Common knowledge of rational expectations imply

$$\sum_{k=0}^{\infty} a_k = -\frac{\delta\gamma\lambda}{1-\lambda\rho} \tag{2}$$

(1)+(2) together with bounded variance of higher order expectations $\implies$ approximation error variance can be made arbitrarily small

$$\lim_{\overline{k}\to\infty} E\left(\mathbf{a}\theta_{t|t}^{(0:\overline{k})} - \mathbf{a}\theta_{t|t}^{(0:\infty)}\right)^2 = 0$$

The law of motion of the expectations hierarchy

We want the form

$$\theta_{t|t}^{(0:\infty)} = M\theta_{t-1|t-1}^{(0:\infty)} + N \begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix}$$

Process for actual state

$$\theta_t = \rho\theta_{t-1} + v_t$$

Trader j's hierarchy updating equation

$$\theta_{t|t}^{(1:\infty)}(j) = M\theta_{t-1|t-1}^{(1:\infty)}(j) + K\left(S_t(j) - LM\theta_{t-1|t-1}^{(1:\infty)}(j) - Qc_t\right)$$

The law of motion of the expectations hierarchy

Signal vector is a function of the state

$$\int S_t(j)\ dj = L\theta_{t|t}^{(0:\infty)} + Qc_t$$

$$L = \begin{bmatrix} e_1' \\ \mathbf{a} \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 \\ \frac{\lambda\psi}{1-\lambda\psi} \end{bmatrix}$$

We can write the average hierarchy updating equation as

$$\theta_{t|t}^{(1:\infty)} = (I - KL)\, M\theta_{t-1|t-1}^{(1:\infty)} + KLM\theta_{t-1|t-1}^{(0:\infty)} + KLN \begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix} + K \begin{bmatrix} 0 \\ -\delta\gamma\lambda\epsilon_t \end{bmatrix}$$

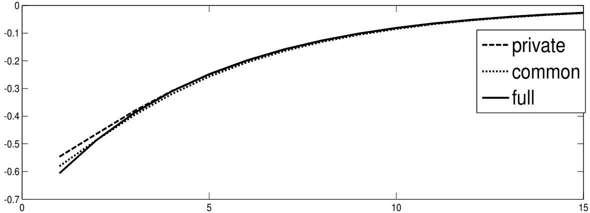
The law of motion of the expectations hierarchy

Actual process and average hierarchy updating equation imply

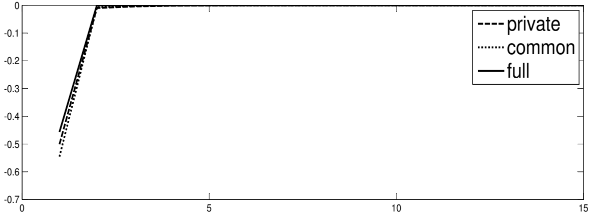
$$\begin{aligned} M &= \begin{bmatrix} \rho & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ KLM \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & (I - KL)\, M \end{bmatrix} \\ N &= \begin{bmatrix} e_1' \\ KLN \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbf{0} & -K_2 \delta \gamma \lambda \end{bmatrix} \end{aligned}$$

To solve the model, find a fixed point for M, N, a and δ

Asset price IRF with Singleton's parameters

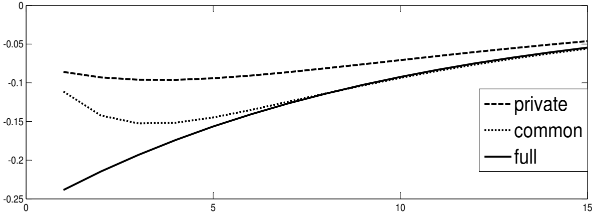


Persistent supply shock

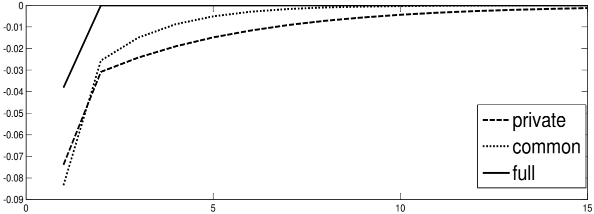


Transitory supply shock

Private information matters



Persistent supply shock



Transitory supply shock

The method in 3 steps

1. Impose structure on higher order expectations through common knowledge of rational expectations
2. Variance of expectations non-increasing with order of expectation
3. Impact of expectations decreasing with order of expectation

Previous strategies

- Lagged revelation of shocks: Townsend (1983), Singleton (1987)
  - Not always realistic and can result in weird (kinked) IRF (Bacchetta and Van Wincoop (2006))
- Finite horizon: Allen, Morris and Shin (2006)
- Static choices: Woodford (2002), Morris and Shin (2002)
- Frequency domain methods: Kasa, Walker and Whiteman (2006)
  - Analytically elegant, but hard to tell how generally applicable method is

The method of Nimark (2007) seems quite useful

- No need to assume lagged shock revelation
- Can handle dynamic choices in infinite horizon models
- Multi dimensional hidden state
  - E.g. Nimark (forthcoming) on pricing decisions in a general equilibrium macro model
- Endogenous hidden state
  - E.g. capital stock in RBC model with incomplete markets and idiosyncratic technology shocks (Graham and Wright 2007)

But I guess I am biased.

More of My-Method-Mongering

- Transparent
  - Explicit modeling of higher order expectations helps intuition
- General
  - Fewer modeling compromises due to private information
- Fast
  - Solution fast enough for empirical work