Modelling Information, Learning and Expectations in Macroeconomics

Lecture 8

New York University

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The informational content of endogenous outcomes

Today:

Can the expectations of other agents be inferred form observing endogenous variables?

When is the state perfectly revealed by observables?

Forecasting the Forecasts of Others

Townsend (JPE 1983) coined the term *Forecasting the Forecasts of Others*
An ironic twist to the history of the topic:

- Subsequent research has shown that in equilibrium, agents share the same forecasts so there is actually no need to "forecast the forecasts of others".

A bit of Townsend model history

Sargent (JEDC 1991)

- Uses a guess and verify strategy of the functional form of the process that firms try to fit observable variables to.
- Discovers numerically that firms in different industries share the same forecasts and capital stocks

Kasa (RED 2000)

- Uses frequency domain methods to analytically solve the model and to show that agents share the same information

Pearlman and Sargent (RED 2005)

- Uses time domain methods to solve the model and show that agents share the same information

All these methods involve quite an effort. What can we know before we solve the model?

Townsend model refresher

Output sector i

$$y_t^i = f_0 k_t^i$$

Market clearing price

$$\begin{aligned} P_t^i &= -b_1 Y_t^i + z_t^i \\ z_t^i &= \theta_t + \epsilon_t^i \end{aligned}$$

where

$$\theta_t = \rho \theta_{t-1} + v_t$$

and $\epsilon_t^i \sim N(0, \sigma_\epsilon^2)$ and $v_t \sim N(0, \sigma_v^2)$

Townsend model refresher

The firms profit max problem:

$$\max_{\{k_t^i\}_{t=1}^{\infty}} E_0^i \sum_{t=0}^{\infty} \beta^t \left[P_t^i f_0 k_t^i - \frac{f_1}{2} \left(k_t^i\right)^2 - \frac{f_2}{2} \left(k_{t+1}^i - k_t^i\right)^2 \right]$$

$$f_0, f_2 \quad > \quad 0, \qquad f_1 \geq 0$$

Decision rule

$$k_{t+1} = \lambda_1 k_t^i + \frac{f_0 \beta \lambda_1}{f_2} \sum_{j=0}^{\infty} (\beta \lambda_1)^j E\left(P_{t+1+j}^i \mid \Omega_t^i\right)$$

gives a law of motion for aggregate industry i capital stock

$$K_{t+1}^i = h_1 K_t^1 + h_2 M_t^i$$

where

$$M_t^i = E\left(\theta_t \mid \Omega_t^i\right)$$

What firms observe

Firms need to form an estimate of θ_t to invest optimally in their capital stock. Firms in industry 1 observes

$$\begin{bmatrix} z_t^1 \\ P_t^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t^1 \\ \epsilon_t^2 \\ \theta_t \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} b_1 f_0 K_t^2$$

Firms in industry 2 observes

$$\begin{bmatrix} P_t^1 \\ z_t^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t^1 \\ \epsilon_t^2 \\ \theta_t \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} b_1 f_0 K_t^1$$

Remember: The only link between industries are that they both try to estimate the same unobservable state

What firms observe

Start with industry 1:

$$
\begin{aligned}
\begin{bmatrix} z_t^1 \\ P_t^2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t^1 \\ \epsilon_t^2 \\ \theta_t \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} b_1 f_0 K_t^2 \\
Z_t^1 &= DX_t - F\left(\left\{Z_{t-s}^2\right\}_{s=1}^{\infty}\right)
\end{aligned}
$$

K_t^2 is predetermined and depend on industry 2 firms' previous estimate of θ_{t-1} and K_{t-1}^2 and thus only on information dated in period t-1 or earlier.

The Projection Theorem

If $\mathcal{M}$ is a closed subspace of the Hilbert Space $\mathcal{H}$ and x$\in \mathcal{H}$, then
(i) there is a unique element $\widehat{x} \in \mathcal{M}$ such that

$$\|x - \widehat{x}\| = \inf_{y \in \mathcal{M}} \|x - y\|$$

and
(ii) $\widehat{x} \in \mathcal{M}$ and $\|x - \widehat{x}\| = \inf_{y \in \mathcal{M}} \|x - y\|$ if and only if $\widehat{x} \in \mathcal{M}$ and $(x - \widehat{x}) \in \mathcal{M}^{\perp}$ where $\mathcal{M}^{\perp}$ is the orthogonal complement to $\mathcal{M}$ in $\mathcal{H}$.
The element $\widehat{x}$ (or $\mathcal{P}_{\mathcal{M}} x$) is called the orthogonal projection of x onto $\mathcal{M}$.

The Space Spanned by Observables

We want to show that $\left\{Z_{t-s}^1\right\}_{s=0}^t$ and $\left\{Z_{t-s}^2\right\}_{s=0}^t$ span the same space.

Define the space $\mathcal{M}^1$ so that $y \in \mathcal{M}^1$ implies

$$
\begin{aligned}
y &= \alpha_0 Z_t^1 + \alpha_1 Z_{t-1}^1 + \ldots + \alpha_t Z_0^1 \\
&= \sum_{s=0}^{t} \alpha_s Z_{t-s}^1
\end{aligned}
$$

for some αs.

Sufficient to show that $y \in \mathcal{M}^1 \implies y \in \mathcal{M}^2$ and $y \in \mathcal{M}^2 \implies y \in \mathcal{M}^1$.

What firms observe

Remember, current observables of Industry 1 is given by

$$Z_t^1 = DX_t - F\left(\left\{Z_{t-s}^2\right\}_{s=1}^{\infty}\right)$$

F is a linear function so we can write the vector of the history of observable variables for Industry 1 as

$$\begin{bmatrix} Z_t^1 \\ Z_{t-1}^1 \\ Z_{t-2}^1 \\ \vdots \\ Z_0^1 \end{bmatrix} = \underbrace{\begin{bmatrix} I & F_1 & F_2 & \cdots & F_{t-1} \\ 0 & I & F_1 & & F_{t-2} \\ 0 & 0 & I & & F_{t-3} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}}_{full\ rank} \begin{bmatrix} DX_t^1 \\ DX_{t-1}^1 \\ DX_{t-2}^1 \\ \vdots \\ DX_{t0}^1 \end{bmatrix}$$

with a symmetric expression holding for Industry 2

The space spanned by observables

Define $\mathbf{Z}_t^i = \begin{bmatrix} Z_t^i & Z_{t-1}^i & \cdots & Z_0^i \end{bmatrix}'$ and $\mathbf{DX}_t = \begin{bmatrix} DX_t & DX_{t-1} & \cdots & DX_0 \end{bmatrix}'$

$$\begin{aligned} \mathbf{Z}_t^1 &= \mathbf{F}^1 \mathbf{DX}_t \\ \mathbf{Z}_t^2 &= \mathbf{F}^2 \mathbf{DX}_t \end{aligned}$$

and

$$\begin{aligned} \mathbf{Z}_t^1 &= \mathbf{F}^1 \left[\mathbf{F}^2\right]^{-1} \mathbf{Z}_t^2 \\ \mathbf{Z}_t^2 &= \mathbf{F}^2 \left[\mathbf{F}^1\right]^{-1} \mathbf{Z}_t^1 \end{aligned}$$

so $\mathbf{Z}_t^1$ and $\mathbf{Z}_t^2$ span the same space $\implies \mathcal{P}_{\mathcal{M}1} x = \mathcal{P}_{\mathcal{M}2} x$

Continuum of Industries

Kasa (2000) also solves model with a continuum of industries but adds a measurement error to the observation of the average price across industries.Firms in industry i observes

$$\begin{bmatrix} z_t^i \\ \overline{P}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t^1 \\ \eta_t \\ \theta_t \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} b_1 f_0 \overline{K}_t$$

Same argument applies again so industries share forecasts but do not know the value of θ_t with certainty.

Continuum of Industries

What if there are no measurement errors η_t?

$$\begin{bmatrix} z_t^i \\ \overline{P}_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t^1 \\ \theta_t \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} b_1 f_0 \overline{K}_t$$

Can we just invert the system to infer the state from observables?

$$\begin{bmatrix} Z_t^1 \\ Z_{t-1}^1 \\ Z_{t-2}^1 \\ \vdots \\ Z_0^1 \end{bmatrix} = \underbrace{\begin{bmatrix} D & F_1 D & F_2 D & \cdots & F_{t-1} D \\ 0 & D & F_1 D & & F_{t-2} D \\ 0 & 0 & D & & F_{t-3} D \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D \end{bmatrix}}_{\textit{full rank}} \begin{bmatrix} X_t^1 \\ X_{t-1}^1 \\ X_{t-2}^1 \\ \vdots \\ X_{t0}^1 \end{bmatrix}$$

Usually yes.

Invertible and non-invertible MA representations

Consider the MA(1) process

$$\begin{aligned} Z_t &= (1+\theta L)\varepsilon_t \\ E\varepsilon_t\varepsilon_{t+s} &= \sigma^2 \text{ for } s=0 \\ &= 0 \text{ otherwise} \end{aligned}$$

We can pre-multiply both sides with $(1+\theta L)^{-1}$

$$(1+\theta L)^{-1} Z_t = \varepsilon_t$$

or

$$\left(1-\theta L-\theta^2 L^2-\theta^3 L^3 \ldots\right) Z_t = \varepsilon_t$$

We need $|\theta|<1$ to hold in order to be able infer the innovation ε_t from the history of Z_t

Does this has interesting implications for economics?

Invertible and non-invertible MA representations

Kasa, Walker and Whiteman (2007) and Walker (JET 2007) says yes. But:
For any non-invertible MA representation there always exist an observationally equivalent invertible representation (see Hamilton 1994)

$$\begin{aligned} Z_t &= (1+\widetilde{\theta}L)\widetilde{\varepsilon}_t \\ \widetilde{\theta} &= \theta^{-1} \\ E\widetilde{\varepsilon}_t\widetilde{\varepsilon}_{t+s} &= \sigma^2\widetilde{\theta}^{-2} \text{ for } s=0 \\ &= 0 \text{ otherwise} \end{aligned}$$

They do not provide economic examples where ε_t matters independently of Z_t though.

Invertible states

Baxter, Graham and Wright (2008) discuss related concepts:

1. Instantaneous invertibility
2. Asymptotic invertibility

Motivated by Mehra and Prescott's (1990) justification for full information rational expectations models that states are *"an invertible function of observables"*

When are states an invertible function of observables?

$$
\begin{aligned}
\underset{(n\times 1)}{X_t} &= \underset{(n\times n)}{A} X_{t-1} + \underset{(n\times m)}{C} \mathbf{u}_t \\
\underset{(l\times 1)}{Z_t} &= D X_t
\end{aligned}
$$

Same as state space system we analyzed before, except measurement errors are now included in the state X_t

When are states an invertible function of observables?

$$\underset{(n\times 1)}{X_t} = \underset{(n\times n)}{A} X_{t-1} + \underset{(n\times m)}{C} \mathbf{u}_t$$
$$\underset{(l\times 1)}{Z_t} = DX_t$$

Trivial cases:

Single entry row in D :

- At least one state variable perfectly observed

Instantaneous invertibility:

- D is square and invertible

When are states an invertible function of observables?

$$\underset{(n\times 1)}{X_t} = \underset{(n\times n)}{A} X_{t-1} + \underset{(n\times m)}{C} \mathbf{u}_t$$

$$\underset{(l\times 1)}{Z_t} = D X_t$$

More interesting if $l < n$ but $l = m$.
Examples:

- RBC models
- DSGE models with pre-determined endogenous states

Asymptotic invertibility

$$\begin{aligned} \underset{(n\times 1)}{X_t} &= \underset{(n\times n)}{A} X_{t-1} + \underset{(n\times m)}{C} \mathbf{u}_t \\ \underset{(l\times 1)}{Z_t} &= DX_t \end{aligned}$$

Baxter, Graham and Wright show that necessary and sufficient conditions for asymptotic invertibility are

1. $l = m$
2. $|DC| \neq 0$
3. $\max |eig \ [(I - K_t D)A]| < 1$

Asymptotic invertibility

To see where the third point $(max\,|eig\;\;[(I-KD)A]| < 1)$ comes from, write the Kalman filter error as

$$\left[X_t - X_{t|t}\right] = (I - K_t D)A\left[X_{t-1} - X_{t-1|t-1}\right] + (I - K_t D)C\mathbf{u}_t$$

since

$$\begin{aligned} X_t &= AX_{t-1} + C\mathbf{u}_t \\ X_{t|t} &= (I - KD)\,AX_{t-1|t-1} + KDX_t \end{aligned}$$

It should be clear that 3. is necessary, but sufficiency of conditions 1-3 is much more involved to prove.

Instantaneous and asymptotic invertibility: The linearised RBC model

$$
\begin{aligned}
k_{t+1} &= \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2)\, c_t \\
a_{t+1} &= \phi a_t + u_{t+1} \\
c_t &= g_1 k_t + g_2 a_t
\end{aligned}
$$

Invertibility of the linearised RBC model

By substituting in the policy function into the law of motion for capital we can put the model in the usual state space form

$$\begin{bmatrix} k_{t+1} \\ a_{t+1} \end{bmatrix} = A \begin{bmatrix} k_t \\ a_t \end{bmatrix} + Cu_{t+1}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} w_t \\ r_t \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \\ -\lambda_3 & \lambda_3 \end{bmatrix} \begin{bmatrix} k_t \\ a_t \end{bmatrix}$$

If we can observe both wages w_t and return on capital r_t and $\lambda_3 \neq 0$ then system is instantaneously invertible

Invertibility of the linearised RBC model

What if we can only observe return on capital?

$$\begin{aligned} \begin{bmatrix} k_{t+1} \\ a_{t+1} \end{bmatrix} &= A \begin{bmatrix} k_t \\ a_t \end{bmatrix} + Cu_{t+1}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ [r_t] &= \begin{bmatrix} -\lambda_3 & \lambda_3 \end{bmatrix} \begin{bmatrix} k_t \\ a_t \end{bmatrix} \end{aligned}$$

Checking conditions 1-3 of Baxter, Graham and Wright is straight forward if $\max |eig(A)| < 1$

Invertibility of the linearised RBC model

We can also check numerically that if we start the Kalman recursion

$$\begin{aligned} & P_{t|t-1} \\ = \; & A\left(P_{t-1|t-2} - P_{t-1|t-2}D'\left(DP_{t-1|t-2}D' + \Sigma_{vv}\right)^{-1} DP_{t-1|t-2}\right)A' \\ & + CC' \end{aligned}$$

from $P_{0|0} = E\left(X_0 - X_{0|0}\right)\left(X_0 - X_{0|0}\right)'$ and conditions 1-3 are satisfied, then

$$\begin{aligned} \lim_{t\to\infty} P_{t|t} &= 0 \\ \lim_{t\to\infty} P_{t|t-1} &= CC' \end{aligned}$$

Let's check.

General take home lesson:

With enough shocks relative to observables, private or imperfect information can always be maintained in equilibrium

- But might be worth checking formally before spending too much effort on solving a model.