TOPICS IN MACROECONOMICS: MODELING INFORMATION, LEARNING AND EXPECTATIONS

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The Information Revealed by Market Prices

In this course we are studying models in which agents do not have full information about the state of the economy. However, most economists are quite content with assuming that agents know the state of the economy with certainty. The justifications for this assumption falls broadly into two categories: 1. Information imperfections do not matter, so we do not loose anything by abstracting from them, or. 2. The assumption is meant literally. To make the first argument, one of course first have to solve imperfect information models and compare their predictions to those of full information models. To motivate the literal interpretation, it is common to assume that the state is an invertible function of prices. That is, markets convey all available information about the state through prices. These notes present a theoretical argument for why we should not expect prices to be perfectly revealing, due to Grossman and Stiglitz (1980). Grossman and Stiglitz' argument also has some support in casual observations as illustrated by the fact that investors pay for surveys of others' expectations. If prices perfectly revealed the state, these surveys would be useless and no one would be willing to pay for them.

Another implication if the state was perfectly revealed by prices would be that all market participants would share the same expectations, but surveys often report quite a spread between different forecasters. If prices revealed the state perfectly, it would be trivial to back

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out the “best” forecast from prices and all surveyed forecasters would agree that indeed, yes, one particular forecast is the optimal one.

1. The Impossibility of Informationally Efficient Markets

Grossman and Stiglitz (1980) presents an elegant argument why markets cannot perfectly reveal all relevant information if information gathering is costly. It is a simple model of a single period asset market with traders with Constant Absolute Risk Aversion (CARA) preferences.

1.1. **The Model.**

There are two assets: A safe asset yielding return R and a risky asset yielding return u

$$u = \theta + \varepsilon \tag{1.1}$$

where θ is observable at cost c and both θ and ε are random variables with the following properties:

$$\begin{aligned} Eu &= \mu_u \\ E\varepsilon &= 0 \\ E\theta\varepsilon &= 0 \\ Var\left(u \mid \theta\right) &= Var\left(\varepsilon\right) \equiv \sigma_\varepsilon^2 > 0 \end{aligned}$$

There are two types of (ex ante identical) agents: Those who observe θ (informed traders) and those who don’t (uninformed traders). That is, ex ante identical agents choose whether or not to pay the cost c to observe the signal θ.

Trader i’s (indirect) utility is given by

$$V\left(W_{1i}\right) = -e^{-aW_{1i,}} \qquad , a > 0 \tag{1.2}$$

where a is the coefficient of absolute risk aversion and W_{1i} is next period wealth, given by

$$W_{1i} = RM_i + uX_i \tag{1.3}$$

where M_i and X_i are trader i's holdings of the safe and risky asset, respectively. We also hav ethe period 0 budget constraint

$$W_{0i} = M_i + PX_i$$

The budget constrintTrader i choose his asset demand X_i to maximize the expected next period utility, given by

$$\begin{aligned} EV\left(W_{1i} \mid I_i\right) &= E\left(-e^{-aW_{1i,}} \mid I_i\right) & (1.4) \\ &= -\exp\left(-a\left\{E\left(W_{1i} \mid I_i\right) - \frac{a}{2}VarE\left(W_{1i} \mid I_i\right)\right\}\right) & (1.5) \end{aligned}$$

where we used that the expectation of a log normal variable is equal to the log of the mean, minus half the conditional variance. The first order condition can be found by substituting in the expression of period 1 wealth (1.3) into the expected utility function (1.2) to get

$$EV\left(W_{1i} \mid I_i\right) = -\exp\left(-a\left[RW_{0i} + X_i\left[E\left(u \mid I_i\right) - RP\right] - \frac{a}{2}X_i^2 E\left[u - E\left(u \mid I_i\right)\right]^2\right]\right) \tag{1.6}$$

Differentiating w.r.t. X_i

$$\frac{\partial EV\left(W_{1i} \mid I_i\right)}{\partial X_i} = -a\left[E\left(u \mid I_i\right) - RP\right] + a^2 X_i E\left[u - E\left(u \mid I_i\right)\right]^2 \tag{1.7}$$

and setting the result equal to zero to get the f.o.c.

$$\frac{\partial EV\left(W_{1i} \mid I_i\right)}{\partial X_i} = 0 \tag{1.8}$$

$$a\left[E\left(u \mid I_i\right) - RP\right] - a^2 X_i E\left[u - E\left(u \mid I_i\right)\right]^2 = 0 \tag{1.9}$$

$$\Longleftrightarrow$$

$$X_i = \frac{\left[E\left(u \mid I_i\right) - RP\right]}{aE\left[u - E\left(u \mid I_i\right)\right]^2} \tag{1.10}$$

We can check that the expression for trader i's demand makes sense: Higher expected return increases demand, higher conditional return variance and risk aversion a decreases demand as does a higher return R on the alternative investment. A useful property that helps keep CARA utility based models tractable is that demand is independent of wealth.

This expression is the same for both informed and uninformed agents. The difference will come from the information sets that these expectations are conditioned on. Informed traders condition on θ and will make conditional expectation errors of u equal to ε with variance σ_ε^2

$$X_I = \frac{E\left(u \mid \theta\right) - RP}{aE\left[u - E\left(u \mid \theta\right)\right]^2} \tag{1.11}$$

$$= \frac{E\left(u \mid \theta\right) - RP}{a\sigma_\varepsilon^2} \tag{1.12}$$

Uninformed traders will condition on the price P and make conditional errors with variance $\sigma_{u|p}^2$

$$X_U = \frac{E\left(u \mid P\right) - RP}{aE\left[u - E\left(u \mid P\right)\right]^2} \tag{1.13}$$

$$= \frac{E\left(u \mid P\right) - RP}{a\sigma_{u|p}^2} \tag{1.14}$$

Equilibrium price (for a given proportion λ of informed traders and given conditional variances) can be found by equating aggregate demand and (exogenous) supply $X \sim N\left(\mu_x, \sigma_x^2\right)$

$$\lambda X_I + (1-\lambda) X_U = X \tag{1.15}$$

$$\lambda \frac{E(u \mid \theta) - RP}{a\sigma_{u|\theta}^2} + (1-\lambda) \frac{E(u \mid P) - RP}{a\sigma_{u|P}^2} = X \tag{1.16}$$

$$\Longleftrightarrow$$

$$\begin{aligned} \lambda \frac{RP}{a\sigma_{u|\theta}^2} + (1-\lambda) \frac{RP}{a\sigma_{u|P}^2} &= -X + \lambda \frac{E(u \mid \theta)}{a\sigma_{u|\theta}^2} \\ &\quad + (1-\lambda) \frac{E(u \mid P)}{a\sigma_{u|P}^2} \end{aligned} \tag{1.17}$$

$$\Longleftrightarrow$$

$$\begin{aligned} P &= \left[\left(\frac{\lambda}{a\sigma_{u|\theta}^2} + \frac{(1-\lambda)}{a\sigma_{u|P}^2}\right) R\right]^{-1} \\ &\quad \times \left[\lambda \frac{E(u \mid \theta)}{a\sigma_{u|\theta}^2} + (1-\lambda) \frac{E(u \mid P)}{a\sigma_{u|P}^2} - X\right] \end{aligned} \tag{1.18}$$

In the case of all traders being informed, i.e. $\lambda = 1$, this simplifies to

$$P_{\lambda=1} = \left[\frac{E(u \mid \theta)}{R} - \frac{a\sigma_{u|\theta}^2}{R} X\right] \tag{1.19}$$

$$= \frac{\theta - a\sigma_\varepsilon^2 X}{R} \tag{1.20}$$

That is, prices increase in expected return but decreases in risk aversion a, conditional variance of the return σ_ε^2 and the return R on the alternative investment (the safe asset).

$$P_{\lambda=0} = \left[\frac{E(u \mid P)}{R} - \frac{a\sigma_{u|P(\lambda=0)}^2}{R} X\right] \tag{1.21}$$

$$= \frac{E(u) - a\left(\sigma_\theta^2 + \sigma_\varepsilon^2\right) X}{R} \tag{1.22}$$

where we used that the price is uninformative about u when nobody buys the signal.

1.2. **Understanding when it pays to switch strategy.** Looking at two limit cases can help intuition for the decision of agents and when an (interior) equilibrium exists. First, note that the decision to buy the signal or not is made ex ante, so below we will have to replace random variables with their expected values.

1.2.1. *Case 1: Everybody is informed.* First, consider an agent in a world where everybody else has chosen to buy the signal θ and paying the cost c. The benefit of the decrease in conditional return variance from observing the signal instead of only observing the price (which can be observed for free) must then be larger than the cost Rc in terms of expected utility. Formally, it will pay to switch strategy if the inequality holds

$$\begin{aligned} EV\left(W_{1i} \mid \theta\right) &= -\exp\left(-a\left[RW_{0i} - cR + E\left(X_{I,\lambda=1}\left[\theta - RP_{\lambda=1}\right]\right) - \frac{a}{2}E\left(X_{I,\lambda=1}^2\sigma_\varepsilon^2\right)\right]\right) \qquad (1.23) \\ &< -\exp\left(-a\left[RW_{0i} + E\left(X_{U,\lambda=1}\left[E\left(u \mid P\right) - RP_{\lambda=1}\right]\right) - \frac{a}{2}E\left(X_{U,\lambda=1}^2\sigma_{u|P(\lambda=1)}^2\right)\right]\right) \end{aligned}$$

The first line is the expected utility of being informed when all traders are informed and the second line is the expected utility of being the only uninformed agent. We can simplify to get

$$\begin{aligned} cR &> E\left[X_{I,\lambda=1}\left(\left[E\left(u\right) - RP_{\lambda=1}\right] - \frac{a}{2}X_{I,\lambda=1}\sigma_\varepsilon^2\right)\right] \\ &\quad -E\left[X_{U,\lambda=1}\left(\left[E\left(u\right) - RP_{\lambda=1}\right] - \frac{a}{2}X_{U,\lambda=1}\sigma_{u|P(\lambda=1)}^2\right)\right] \end{aligned} \qquad (1.24)$$

where we used that the ex ante, the unconditional and conditional return expectation are the same (that is, the choice to buy the signal or not is made before the signal is observed). Obviously, for bounded conditional variances, there exists a c large enough to make the inequality (1.24) hold. However, to check for the existence of an interior equilibrium we want to find an interval over which c is large enough to make it profitable to not buy information when everybody else is doing so, but small enough to make it worthwhile when no one else is buying information. To do so we need to look at the opposite limit.

1.2.2. *Case 2: Everybody is uninformed.* If everybody is uninformed, we need that the expected utility of switching to being informed outweighs the cost of the signal so that the inequality

$$\begin{aligned} EV\left(W_{1i} \mid \theta\right) &= -\exp\left(-a\left[RW_{0i} - cR + E\left(X_{I,\lambda=0}\left[\theta - RP_{\lambda=0}\right]\right) - \frac{a}{2}X_{I,\lambda=0}^{2}\sigma_{\varepsilon}^{2}\right]\right) \\ &> -\exp\left(-a\left[RW_{0i} + E\left(X_{U,\lambda=0}\left[E\left(u \mid P\right) - RP_{\lambda=0}\right]\right) - \frac{a}{2}X_{U,\lambda=0}^{2}\sigma_{u|P(\lambda=1)}^{2}\right]\right) \end{aligned} \tag{1.25}$$

must hold for some c for there to exist an interior solution. Again, rearranging and simplifying yields

$$\begin{aligned} cR &< E\left[X_{I,\lambda=0}\left(\left[E\left(u\right) - RP_{\lambda=0}\right] - \frac{a}{2}X_{I,\lambda=0}\sigma_{\varepsilon}^{2}\right)\right] \\ &\quad -E\left[X_{U,\lambda=0}\left(\left[E\left(u\right) - RP_{\lambda=0}\right] - \frac{a}{2}X_{U,\lambda=0}\sigma_{u|P(\lambda=0)}^{2}\right)\right] \end{aligned} \tag{1.26}$$

That is, there must exists a cost smaller than the benefit of buying the signal when nobody else does so. For an interior solution to exists, there must be a cost small enough for the marginal trader to deviate from the strategy of everybody else at both limit points, i.e. both to buy the signal when nobody else does as well as not buy the signal when everybody else does.

We can think of the right hand side of (1.24) as an lower bound $\underline{c}R$ on the interval and the right hand side of (1.26) as the upper bound $\overline{c}R$ on the interval of costs that will yield interior solutions. (If $\overline{c} < \underline{c}$ no interior solution exists.)

We could verify the inequalities above numerically by computing the conditional variances $\sigma^2_{u|P(\lambda=0)}$ and $\sigma^2_{u|P(\lambda=1)}$. This can be done by exploiting the following relationships

$$E\left(P_{\lambda=1}\right) = \frac{E(u)}{R} - \frac{a\sigma_\varepsilon^2}{R}E\left(X\right) \tag{1.27}$$

$$E\left(P_{\lambda=0}\right) = \frac{E(u)}{R} - \frac{a\left(\sigma_\theta^2+\sigma_\varepsilon^2\right)}{R}E\left(X\right) \tag{1.28}$$

$$E\left(X_{I,\lambda=1}\right) = \frac{E\left(u\right)}{a\sigma_\varepsilon^2} - \frac{R}{a\sigma_\varepsilon^2}\left(\frac{E(u)}{R} - \frac{a\sigma_\varepsilon^2}{R}E\left(X\right)\right) = E\left(X\right) \tag{1.29}$$

$$E\left(X_{U,\lambda=1}\right) = \frac{E\left(u\right)}{a\sigma^2_{u|P(\lambda=1)}} - \frac{R}{a\sigma^2_{u|P(\lambda=1)}}\left(\frac{E(u)}{R} - \frac{a\sigma_\varepsilon^2}{R}E\left(X\right)\right) = \frac{\sigma_\varepsilon^2}{\sigma^2_{u|P(\lambda=1)}}E\left(X\right) \tag{1.30}$$

$$E\left(X_{I,\lambda=0}\right) = \frac{E\left(u\right)}{a\sigma_\varepsilon^2} - \frac{R}{a\sigma_\varepsilon^2}\left(\frac{E(u)}{R} - \frac{a\sigma^2_{u|P(\lambda=0)}}{R}E\left(X\right)\right) = \frac{\sigma^2_{u|P(\lambda=0)}}{\sigma_\varepsilon^2}E\left(X\right) \tag{1.31}$$

$$E\left(X_{U,\lambda=0}\right) = E\left(X\right) \tag{1.32}$$

together with the fact that the conditional variance of joint normally distributed variables Y and X are given by

$$E\left[E\left(Y \mid X\right) - Y\right]^2 = E\left(YY'\right) - E\left(YX'\right)E\left(XX\right)^{-1}E(XY') \tag{1.33}$$

Using this formula and the expression for the price, the variance of returns conditional on $P_{\lambda=1}$ is then given by

$$E\left[E\left(u \mid P_{\lambda=1}\right) - u\right]^2 = E\left(u^2\right) - E\left(uP_{\lambda=1}\right)E\left(P_{\lambda=1}P_{\lambda=1}\right)^{-1}E(P_{\lambda=1}u) \tag{1.34}$$

$$= \left(\sigma_\theta^2+\sigma_\varepsilon^2\right) - \sigma_\theta^2R^{-1}\left(\sigma_\theta^2R^{-2} + \left(a\sigma_\varepsilon^2\right)^2\sigma_x^2R^{-2}\right)^{-1}R^{-1}\sigma_\theta^2 \tag{1.35}$$

$$= \left(\sigma_\theta^2+\sigma_\varepsilon^2\right) - \frac{\left(\sigma_\theta^2\right)^2}{\sigma_\theta^2 + \left(a\sigma_\varepsilon^2\right)^2\sigma_x^2} \tag{1.36}$$

and the variance of returns conditional on $P_{\lambda=0}$ are simply

$$E\left[E\left(u \mid P_{\lambda=0}\right) - u\right]^2 = \left(\sigma_\theta^2+\sigma_u^2\right) \tag{1.37}$$

since $E\left(uP_{\lambda=0}\right) = 0$.

1.3. **Grossman and Stiglitz impossibility argument.** It is now hopefully straightforward to understand the main argument of Grossman and Stiglitz's paper: Prices cannot be fully revealing if there is a positive cost c of observing the signal θ. To see why, consider the equilibrium conditional that expected utility of being informed must be the same as the expected utility of being uninformed

$$\frac{-\exp\left(-a\left[RW_{0i} - cR + X_I\left[E\left(u \mid \theta\right) - RP\right] - \frac{a}{2}X_I\sigma_\varepsilon^2\right]\right)}{-\exp\left(-a\left[RW_{0i} + X_U\left[E\left(u \mid P\right) - RP\right] - \frac{a}{2}X_U\sigma_{u|P}^2\right]\right)} = 1 \tag{1.38}$$

If prices are perfectly revealing, allocations and conditional expectations and variances will be identical across informed and uninformed agents. That is

$$X_I = X_U \tag{1.39}$$

$$E\left(u \mid \theta\right) = E\left(u \mid P\right) \tag{1.40}$$

$$\sigma_\varepsilon^2 = \sigma_{u|P}^2 \tag{1.41}$$

must hold if prices are perfectly revealing of the information in the signal θ. But since there is no benefit from observing the signal θ no one would be willing to pay a cost to do so, and we can only have a perfectly revealing equilibrium if information is for free, that is $c = 0$. Another way of seeing this more clearly (at least algebraically) is to define K

$$K \equiv RW_{0i} + X_I\left[E\left(u \mid \theta\right) - RP\right] - \frac{a}{2}X_I^2\sigma_\varepsilon^2 \tag{1.42}$$

$$= RW_{0i} + X_U\left[E\left(u \mid P\right) - RP\right] - \frac{a}{2}X_U^2\sigma_{u|P}^2 \tag{1.43}$$

so that the equilibrium condition (1.37) under fully revealing prices can be written as

$$\frac{-K + cR}{-K} = 1 \tag{1.44}$$

which obviously cannot hold for any $cR > 0$.

2. Mechanical conditions to check if a system is perfectly revealing

The following results from Baxter, Graham and Wright (JEDC 2010) can be useful:

Consider the linear system of the form

$$\underset{(n\times 1)}{X_t} = AX_{t-1} + B\underset{(m\times 1)}{\mathbf{w}_t} \quad : \mathbf{w}_t \sim N(0,\mathbf{I}) \tag{2.1}$$

$$\underset{(s\times 1)}{Z_t} = DX_t \tag{2.2}$$

Definition 1. *The information set* $I_t \equiv \{Z_t, Z_{t-1}, ..., Z_0\}$ *is said to be instantaneously invertible if* $rank(D) = n$ *so that* D *is invertible.*

If the information set I_t is instantaneously invertible, the last observation is sufficient to extract a perfect estimate of the state.

Definition 2. *The information set* $I_t \equiv \{Z_t, Z_{t-1}, ..., Z_0\}$ *is said to be asymptotically invertible iff*

(1) $rank(B) = s$
(2) DB' *is invertible*
(3) $\left|eig\left[\left(I - B\left(DB\right)^{-1} D\right)\right] A\right| < \mathbf{1}$

If these condition are satisfied, the posterior state uncertainty

$$P_{t|t} \equiv E\left(X_t - E\left[X_t \mid I_t\right]\right)\left(X_t - E\left[X_t \mid I_t\right]\right)' \tag{2.3}$$

tend to zero as t increases. That is, $\lim_{t\to\infty} P_{t|t} = \mathbf{0}$. If conditions (1) -(3) are not satisfied, the state X_t can never be perfectly recovered from Z^tso that $\lim_{t\to\infty} P_{t|t} \neq \mathbf{0}$

2.1. A rule-of-thumb.

A rule of thumb which in most cases gives the same answer as the formal condition above is to count shocks, including both structural innovations and measurement errors. If this is a larger number than the number of observable variables, the state will not be perfectly revealed.

2.2. **An easy check for models that are difficult to solve.** If the imperfect information solution is hard to compute and you want to check whether equilibrium outcomes will reveal the state before you go through the trouble of solving the model, there is a way: if the model is easy to solve under perfect information, you can check if the state is an invertible function of equilibrium outcomes in the perfect information solution to the model. The reason this works is that if equilibrium outcomes reveals the state perfectly, the equilibrium in question must be the perfect information equilibrium. So this strategy allow you to check if the model you have in mind will support information imperfections in equilibrium without solving for an imperfect information equilibrium. But the rule-of-thumb is of course even easier.

References

[1] Baxter, Graham and Wright, 2007, *Invertible and non-invertible information sets in dynamic stochastic general equilibrium*,forthcoming, Journal of Economic Dynamics and Control.

[2] Grossman and Stiglitz, 1980, *On the impossibility of informationally efficient markets, American Economic Review.*