TOPICS IN MACROECONOMICS: MODELLING INFORMATION, LEARNING AND EXPECTATIONS

LECTURE NOTES

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Lucas Island Model

The Lucas Island model appeared in a series of papers in the early 1970s (see Lucas 1972, 1973, 1975). These papers became influential for several reasons; they demonstrated the "natural" rate hypothesis and the (long run) neutrality of money in a rigorous setting, they popularized rational expectations and they changed the style of macro economics by building aggregate models on a foundation of optimizing behavior by individuals. What is not always remembered is that business cycles in the island model stems from individuals misperceptions about relative prices, an idea that was almost dead for 20 years from the early 80s to the early 00's. Recently, there has been a revival of the idea that the dynamics of business cycle fluctuations can be influenced by information imperfections, e.g. Mankiw and Reis (2002), Woodford (2002), Mackowiack and Wiederholt (2008) and Lorenzoni (2007).

The models in Lucas's papers differ somewhat in set ups and complexity, and here we will focus on perhaps the most accessible version, that of the 1973 AER paper. The basic idea is the following. Supply (and production) is determined by expected relative prices; when producers expect a high relative price of the good produced on their island, they produce more of it. However, supply decisions are made based on partial information. The nominal price of the good produced on each island is observed only on that island, and the aggregate price level is observed only with a lag. The problem facing the producers on each island is

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thus to figure out how much of the change in their own good's price reflects a general price change and how much reflects a change in relative prices?

Lucas used a prediction from the model to test his hypothesis: in countries where nominal demand varies a lot, producers are more likely to attribute changes in prices to aggregate changes rather than relative price changes. This observation has implications for regression coefficients that are also born out in his empirical exercise.

Here is the model.

0.1. **Supply.** There is a continuum of islands indexed by z. Supply on island z is given by

$$y_t(z) = y_{nt} + y_{ct}(z) \tag{0.1}$$

where y_{nt} is the natural, or trend, component of supply and $y_{ct}(z)$ is the cyclical component. The trend component follows the deterministic law of motion

$$y_{nt} = \alpha + \beta t \tag{0.2}$$

We will mostly be concerned with the cyclical component of supply. Supply on island z is given by

$$y_{ct}(z) = \gamma \left[P_t(z) - E\left(P_t \mid I_t(z)\right)\right] + \lambda y_{c,t-1}(z) \tag{0.3}$$

Lucas sets up the model to make the signal extraction as simple as possible: he assumes that lagged price levels and aggregate output deviations from trend are observable. This result in a common prior for the mean and variance of the aggregate price level

$$P_t \sim N\left(\overline{P}_t, \sigma^2\right) \tag{0.4}$$

The price in island z is (exogenously) given by

$$P_t(z) = P_t + z \tag{0.5}$$

where $z \sim N(0, \tau^2)$. The supply of good z is then given by

$$y_{ct}(z) = \gamma\theta\left[P_t(z) - \overline{P}_t\right] + \lambda y_{c,t-1}(z) \tag{0.6}$$

where

$$\theta = \frac{\tau^2}{\tau^2 + \sigma^2} \tag{0.7}$$

and

$$\sigma^2 = E\left[P_t - \overline{P}_t\right]^2 \tag{0.8}$$

The intuition behind the supply curve (0.6) is that production will be higher on islands with a high expected relative price compared to the aggregate price level.

0.2. **Demand.** (The log of) nominal demand is postulated as

$$y_t + P_t = x_t \tag{0.9}$$

$$\Delta x_t \equiv (x_t - x_{t-1}) \sim N\left(\delta, \sigma_x^2\right) \tag{0.10}$$

0.3. **Information sets.** Lucas assumes that all agents observe the complete history of the aggregate price level and past shocks to demand up to the last period. The only current information available to island z is the price of the good produced on their own island, $P(z)$. The information set at time t on island z is thus given by

$$I_t(z) = \{p_{t-s+1}(z), P_{t-s}, x_{t-s}, y_{t-s} : s = 1, 2, ...\} \tag{0.11}$$

0.4. **Solving the model.** All the action in the model comes from agents misperceptions about the relative price of the good produced on their own island. Without loosing much of interest, we can therefore assume that all constants and deterministic variables are zero, i.e. that $\delta = y_{nt} = 0$, and that λ, the parameter governing the importance of lagged supply, is zero since these only affect the predictable components of output and prices. What remains are the following relationships: An aggregate supply schedule

$$y_t = \theta\gamma \left[P_t - \overline{P}_t\right] \tag{0.12}$$

which can be found by averaging across island specific demand schedules. Aggregate (real) demand can be found by rearranging (0.9) to get

$$y_t = x_t - P_t \tag{0.13}$$

Then combine supply (0.12) and demand (0.13) to get

$$x_t - P_t = \theta\gamma \left[P_t - \overline{P}_t\right] \tag{0.14}$$

To solve the model, we start by conjecturing a solution to P_t and $\overline{P}_t$ of the form

$$P_t = \pi_1 x_t + \pi_2 x_{t-1} \tag{0.15}$$

and

$$\overline{P}_t = \pi_1 E_{t-1}\left[x_t\right] + \pi_2 x_{t-1} \tag{0.16}$$

$$= \pi_1 x_{t-1} + \pi_2 x_{t-1} \tag{0.17}$$

Use (0.15) and (0.16) to substitute out P_t and $\overline{P}_t$ from (0.14)

$$x_t - (\pi_1 x_t + \pi_2 x_{t-1}) = \theta\gamma\left(\pi_1 x_t + \pi_2 x_{t-1}\right) - \theta\gamma(\pi_1 + \pi_2)x_{t-1} \tag{0.18}$$

Equating coefficients then gives

$$1 - \pi_1 = \theta\gamma\pi_1 \tag{0.19}$$

and

$$-\pi_2 = \theta\gamma\pi_2 - \theta\gamma(\pi_1 + \pi_2). \tag{0.20}$$

Solve for π_1 and π_2

$$\pi_1 = \frac{1}{1+\theta\gamma} \tag{0.21}$$

$$\pi_2 = \frac{\theta\gamma}{1+\theta\gamma} \tag{0.22}$$

We can then plug these expressions back into the conjectured solutions

$$P_t = \frac{1}{1+\theta\gamma}x_t + \frac{\theta\gamma}{1+\theta\gamma}x_{t-1} \tag{0.23}$$

$$\overline{P}_t = \left(\frac{1}{1+\theta\gamma} + \frac{\theta\gamma}{1+\theta\gamma}\right)x_{t-1} \tag{0.24}$$

Substituting in the result in the supply curve (0.12)

$$y_t = \theta\gamma\left[P_t - \overline{P}_t\right] \tag{0.25}$$

$$= \frac{\theta\gamma}{1+\theta\gamma}\Delta x_t \tag{0.26}$$

This is not quite a closed form solution of the model, since θ is a function of σ^2

$$\theta = \frac{\tau^2}{\tau^2+\sigma^2} \tag{0.27}$$

and where $\sigma^2 \equiv E\left[P_t - \overline{P}_t\right]^2$ in turn is a function of θ

$$\sigma^2 = \frac{1}{(1+\theta\gamma)^2}\sigma_x^2 \tag{0.28}$$

$$= \frac{1}{\left(1+\frac{\tau^2}{\tau^2+\sigma^2}\gamma\right)^2}\sigma_x^2 \tag{0.29}$$

No explicit solution to this fixed point available, and a solution to (0.28) has to be found numerically. However,but two (analytical) limit cases are still informative.

0.5. **The variance of real output.** The variance of real output $E\left[y_{ct}y_{ct}\right]$ is given by

$$E\left[y_{ct}y_{ct}\right] = \left[\frac{\theta\gamma}{1+\theta\gamma}\right]^2\sigma_x^2$$

Two limit cases:

- If we let $\tau^2 \to 0$ so that $\theta \to 0$ then $E\left[y_{ct}y_{ct}\right] \to 0$, i.e. when the island specific price is a perfect signal of the aggregate price level, real aggregate output volatility tend to zero. This is intuitive, as the only source of output fluctuations in the model are coming from firms who confound relative and nominal price changes.
- If we let $\tau^2 \to \infty$ so that $\theta \to 1$ then $E\left[y_{ct}y_{ct}\right] \to \left[\frac{\gamma}{1+\gamma}\right]^2 \sigma_x^2$ i.e. when the island specific prices are sufficiently volatile relative to the aggregate price level, firms will interpret all price changes as changes in relative prices. Though not immediately clear from (0.27) and (0.28), θ is actually monotonically increasing in τ^2 so that larger variances of Island specific price shocks always lead to more volatile real output.

0.6. **Testable properties of the model.** Changes in the log price level, i.e. inflation, is given by

$$\Delta P_t = \frac{1}{1+\theta\gamma}\Delta x_t + \frac{\theta\gamma}{1+\theta\gamma}\Delta x_{t-1} \tag{0.30}$$

$$= (1-\pi)\,\Delta x_t + \pi\Delta x_{t-1} \tag{0.31}$$

where

$$\pi = \frac{\tau^2\gamma}{(1-\pi)^2\,\sigma_x^2 + \tau^2\,(1+\gamma)} \tag{0.32}$$

from the definition (0.27) of θ. The significance of this expression is that it predicts that π should be small for countries where the variance $\sigma_x^2$of changes in nominal demand Δx_t is high, since π tends monotonically towards zero as σ_x^2 gets larger.

Lucas runs the regressions

$$\Delta P_t = \beta_1\Delta x_t + \beta_2\Delta x_{t-1} + \varepsilon_t^p \tag{0.33}$$

$$y_{ct} = \beta_3\Delta x_t + \varepsilon_t^y \tag{0.34}$$

and confirms the model's predictions that β_2 and β_3 should be a small number for countries with highly variable nominal output.

References

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