TOPICS IN MACROECONOMICS: MODELLING INFORMATION, LEARNING AND EXPECTATIONS

MIDTERM OCTOBER 30 2008 W/SKETCHES OF ANSWERS

There are 4 questions. Question 1 should be answered by everybody. You should choose 2 more questions to answer from the remaining 3. Each answered question is worth a maximum of 10 points. Put your name on each sheet of paper that you hand in. Write clearly. Number the pages. Include a "front page" with your name, which questions you answered and how many pages you handed in. Sign the front page. Good luck.

Question 1: Basic methods, short answers welcome

(1) The variable Y_t is a linear function of the random variable $X_t \sim N(0, \Sigma_{xx})$

$$Y_t = GX_t \tag{0.1}$$

What is the variance of Y_t? **Answer:** $G\Sigma_{xx}G'$

(2) Define $X_{t|t}$ as the linear minimum variance estimate of X_t. What is $E\left[\left(X_t - X_{t|t}\right)X'_{t|t}\right]$?
Answer: 0, or the inner product of the projection error and the projection (both correct, only one needed).

(3) Using the definitions in 1 and 2, which direction of inequality should replace the question mark in

$$E(GX_t)^2 \quad ? \quad E(GX_{t|t})^2 \tag{0.2}$$

and why? When can the inequality be replaced with an equality? **Answer:** $E(GX_t)^2 > E(GX_{t|t})^2$ since $X_t \equiv X_{t|t} + e_t$ where e_t is the estimation error and by optimality it is orthogonal to $X_{t|t}$ so $E(GX_t)^2 = E(GX_{t|t})^2 + E\left(e_t e'_t\right)$. The inequality follows from $E\left(e_t e'_t\right) \geq \mathbf{0}$. The inequality can be replaced with an equality if the estimation errors are zero $E\left(e_t e'_t\right) = 0$ (or trivially, and much less interesting though correct, if $E(GX_t)^2 = 0$).

(4) Let $\mathcal{P}(X|Z,Y)$ denote the orthogonal projection of X on the space spanned by Z and Y. By example, show that

$$\mathcal{P}(X|Z,Y) = \mathcal{P}(X|Z) + \mathcal{P}(X|Y) \tag{0.3}$$

if $\mathcal{P}(Y|Z) = 0$.**Answer:**

$$\begin{aligned}
\mathcal{P}(X|Z,Y) &= E\left(X\begin{bmatrix} Z \\ Y \end{bmatrix}'\right)\left(E\begin{bmatrix} Z \\ Y \end{bmatrix}\begin{bmatrix} Z \\ Y \end{bmatrix}'\right)^{-1}\begin{bmatrix} Z \\ Y \end{bmatrix} \\
&= E\left(\begin{bmatrix} XZ' & XY' \end{bmatrix}'\right)\begin{bmatrix} E[ZZ']^{-1} & 0 \\ 0 & E[ZZ']^{-1} \end{bmatrix}\begin{bmatrix} Z \\ Y \end{bmatrix}
\end{aligned}$$

since $\mathcal{P}(Y|Z) = 0 \implies E[ZY'] = E[YZ'] = 0$. We then have

$$\begin{aligned}
\mathcal{P}(X|Z,Y) &= E[XZ']\,E[ZZ']^{-1}Z + E[XY']\,E[YY']^{-1}Y \qquad (0.4) \\
&= \mathcal{P}(X|Z) + \mathcal{P}(X|Y) \qquad (0.5)
\end{aligned}$$

(5) Let $h(x)$ denote the entropy of the random variable x which has a correlation coefficient with the variable y of 1. What is the mutual information of x and y? **Answer:** Mutual information $I(x;y)$ is defined as

$$I(x;y) = h(x) - h(x \mid y)$$

Perfect correlation between x and y implies a zero conditional entropy $h(x \mid y)$ so that

$$I(x;y) = h(x).$$

Question 2: Lucas (AER 1973) Island Model

Consider island z's (simplified) supply function

$$y_t(z) = \gamma \left[P_t(z) - E\left(P_t\right) \right] \tag{0.6}$$

where the aggregate price level is normally distributed with mean $\overline{P}_t$ and variance σ^2. The price on island z is (exogenously) given by

$$P_t(z) = P_t + z \tag{0.7}$$

where $z \sim N(0, \tau^2)$.

(1) Take $\overline{P}_t$, σ^2 and τ^2 as given. Find an aggregate supply schedule as a function of P_t, $\overline{P}_t, \sigma^2$ and τ^2. **Answer (4 points):** Note that $E\left(P_t\right)$ should be $E\left(P_t \mid P_t(z), \overline{P}_t\right)$. We then have

$$\begin{aligned} y_t(z) &= \gamma \left[P_t(z) - E\left(P_t\right) \right] && (0.8) \\ &= \gamma \left[P_t(z) - (1-\theta)\left(P_t(z) - \overline{P}_t\right) + \theta \overline{P}_t \right] && (0.9) \\ y_t &= \gamma\theta \left[P_t(z) - \overline{P}_t \right] && (0.10) \end{aligned}$$

where

$$\theta = \frac{\tau^2}{\tau^2 + \sigma^2} \tag{0.11}$$

take averages across islands to get aggregate supply schedule

$$y_t = \gamma\theta \left[P_t - \overline{P}_t \right] \tag{0.12}$$

(2) How does higher price level variance σ^2 affect the volatility of output? Use numerical example of σ^2 increasing from 1 to 2 and $\tau^2 = 1$. Discuss intuition. **Answer (3**

points):

$$\sigma_y^2 = (\gamma\theta)^2 \sigma^2 \tag{0.13}$$

$$= \gamma^2 \left(\frac{\tau^2}{\tau^2 + \sigma^2}\right)^2 \sigma^2 \tag{0.14}$$

so for $\sigma^2 = 1$ we have that $\sigma_y^2 = \gamma^2 \left(\frac{1}{2}\right)^2 = \gamma^2 \left(\frac{1}{4}\right)$ and for $\sigma^2 = 2$ we have that $\sigma_y^2 = 2\gamma^2 \left(\frac{1}{3}\right)^2 = \gamma^2 \left(\frac{2}{9}\right)$. The variance of output is thus decreasing in the conditional variance of the price level. There are two opposing forces: On the one hand, output volatility should be increasing since output deviations from the mean are a linear function of the surprise in the price level. However, there is a second effect going in the other direction. That is, less weight is put on the island specific price when the aggregate price level is more volatile. The second effect dominates. Some people derived output variance as a function of σ_x^2 (as in it is in Lucas paper), which is also ok if done correctly.

(3) Explain the empirical exercise Lucas performs to support his theoretical model. **Answer (3 points):** Lucas model predicts that in countries with highly variable nominal demand, inflation should be less correlated with cyclical movements in output as compared to countries where nominal output is less variable. This is because in countries with highly volatile nominal demand, firms on a one island rationally attributes changes in their island's price to changes in the aggregate price level, and not to changes in in the relative price of the good that they produce. Of course, real supply only depends on expected relative prices and not on the nominal price level. Lucas predictions was confirmed in the data, with Argentina being the most conspicuous high nominal demand variance country and the US being an example of a low nominal demand variance country. Some people framed their answer using Lucas' regression equations, which was good, but not necessary.

Question 3: Private and public information

(1) Consider the unobservable variable θ given by

$$\theta \sim N\left(0, \sigma_\theta^2\right) \tag{0.15}$$

Agents (indexed by j) observe a private noisy signal of θ given by

$$\begin{aligned} z(j) &= \theta + \varepsilon(j) \\ \eta(j) &\sim N\left(0, \sigma_\varepsilon^2\right) \forall j \end{aligned} \tag{0.16}$$

That is, all agents receive an equally precise signal of θ but agent j only observes his own signal $z(j)$.Define

$$\theta^{(k)} \equiv \int E\left[\theta^{(k-1)} \mid z(j)\right] \, dj \tag{0.17}$$

$$\theta^{(0)} \equiv \theta \tag{0.18}$$

Find an expression for $\theta^{(k)}$. What is the limit as $k \to \infty$? **Answer (2 points):**

$$\theta^{(k)} = \left(\frac{\sigma_\theta^2}{\sigma_\varepsilon^2 + \sigma_\theta^2}\right)^k \theta \tag{0.19}$$

the limit $\lim_{k\to\infty} \theta^{(k)}$ is zero since $0 < \frac{\sigma_\theta^2}{\sigma_\varepsilon^2+\sigma_\theta^2} < 1$

(2) Explain the role of assuming common knowledge of rationality (i.e. that everybody knows that everybody knows,...and so on,... that everybody forms expectations rationally) in your derivations of part (1) of this question. **Answer (2 points):** Assuming common knowledge of rationality allows us to write down second and higher order expectations as a function of the state of the model, since first (and higher)order expectations then can be treated as any other linear function of the state. A k order expectation is then recursively defined as the (average) rational expectation of the random variable defined as the average $k-1$ order expectations.

(3) Consider the model of Morris and Shin (AER 2002). Utility of agent $i \in (0,1)$ is given by

$$U_i = -(1-r)(a_i - \theta)^2 - r\left(L_i - \overline{L}\right) \tag{0.20}$$

where a_i is the action taken by agent i and

$$L_i = \int (a_j - a_i)^2 \, dj \tag{0.21}$$

and

$$\overline{L} = \int L_j dj \tag{0.22}$$

Agents observe two signals of θ. The public signal y

$$\begin{aligned} y &= \theta + \eta \\ \eta &\sim N(0, \sigma_\eta^2) \end{aligned} \tag{0.23}$$

and the private signal x_i

$$\begin{aligned} x_i &= \theta + \varepsilon_i \\ \varepsilon_i &\sim N(0, \sigma_\varepsilon^2) \forall i \end{aligned} \tag{0.24}$$

The first order condition for expected utility maximization is given by

$$a_i = (1-r) E[\theta \mid x_i, y] + rE\left[\int a_j dj \mid x_i, y\right] \tag{0.25}$$

where $\int a_j dj$ $(\equiv \overline{a})$ is the average action across agents. Find κ in the optimal linear reaction function of agent i

$$a_i = \kappa x_i + (1-\kappa) y \tag{0.26}$$

Solve for equilibrium average action $\overline{a}$ as a function of θ and η. **Answer (4 points):** Exactly as in Lecture Notes 5.

(4) With $\sigma_\eta^2 = \sigma_\varepsilon^2 = 1$, discuss what happens to average actions as functions of θ and η when r switches from 0.5 to -0.5. What is the intuition? **Answer (2 points):** Less weight is put on the public signal (and the public noise η) since actions are now strategic substitutes, i..e. agents get utility from taking actions that are different from the actions taken by other agents. The coordination effect of the public signal is therefore diminished.

Question 4: The information revealed by prices

All questions here are related to the paper *On the impossibility of informationally efficient markets* by Grossman and Stiglitz (AER 1980).

(1) Explain intuitively Grossman and Stiglitz's result regarding the impossibility of informationally efficient markets when information is costly. Carefully define all terminology used. (You may also use some algebra, if you think that would make your answer more accurate.) **Answer (4 points):** If prices convey the same information about returns as a costly signal, no one would be willing to pay for the signal. If nobody wants to buy the signal, prices cannot convey any information about the signal.

(2) Explain intuitively (or with a little algebra) the bounds on the costs of information that guarantees that some agents will and some agents will not buy the signal, i.e.the bounds that guarantees an interior solution.**Answer (3 points):** As in Lecture Notes 6.

(3) Consider an equilibrium where some agents are informed and some are uninformed and where prices do not perfectly reveal the information in the costly signal. Are informed or uninformed agents better off a) ex ante? b) ex post?**Answer (3 points):** The equilibrium condition states that informed and uninformed agents are equally well off ex ante, i.e. they have the same expected utility. Ex post, after signal and returns are realized, there are 4 possibilities and what matters is whether the return

is higher than the safe return and whether the signal indicated that returns would be higher or lower than the safe return. **1. High returns, high signal:** Informed are better off since they put more weight on the positive signal and therefore will hold more of the risky asset (which turned out to have a high return). **2. High returns, low signal:** Informed agents will if the is low enough hold less of the risky asset than the uninformed since they again put more weight on the their more precise signal. Uninformed are better off. **3. Low returns, low signal:** Informed agents hold less of risky asset and is better off ex post. **4. Low returns, high signal:** Informed agents hold more of asset and is worse off