Modeling Information, Learning and Expectations in Macroeconomics

Universitat Pompeu Fabra

Fall 2010

Private/heterogenous/dispersed information

Today:

- *Forecasting the Forecasts of Others* (Townsend JPE 1983)
- *Dynamic Higher Order Expectations* (Nimark WP 2010)

Private/heterogenous/dispersed information

- Every agent has his own "window to the world" but no agent is better informed than others on average
- A framework to think about disagreement and uncertainty about the plans and actions of other agents

Is it important?

- We need to solve quantitative models to answer that question
- The principal modeling difficulty: The infinite regress of "forecasting the forecasts of others" (Townsend 1983)

Solution strategies for models with private information

- Lagged revelation of shocks: Townsend (1983), Singleton (1987)
  - Not always realistic and can result in weird (kinked) IRF (Bacchetta and Van Wincoop (2006))
- Finite horizon: Allen, Morris and Shin (2006)
- Static choices: Woodford (2002), Morris and Shin (2002), Angeletos and La'o (2009)

Nimark (2010) shows how to solve models in which these assumptions are relaxed

The infinite regress of expectations: It's your choice!

Remember from last time: We could solve the Morris and Shin (2002) model with our without an explicit expression for higher order expectations:

- Either predict what you really care about

$$
\begin{aligned}
a_i &= (1-r)\,E\left[\theta \mid I(i)\right] + rE\left[\bar{a} \mid I(i)\right] \\
a_i &= \kappa x_i + (1-\kappa)y
\end{aligned}
$$

- or write an equivalent expression with an explicit role for higher order expectations

$$
\begin{aligned}
a_i &= (1-r)\sum_{k=1}^{\infty} r^{k-1}\theta^{(k)}(i) \\
\theta^{(k)} &= g^k(\theta - y) + y
\end{aligned}
$$

We will discuss both approaches in a dynamic setting today

Some history

- Townsend's paper the first to recognize the infinite regress of expectations in a macro setting, though it was already well-known within game theory.
- Solution method:
  - Make assumption about revelation of state to make relevant history finite.
- The model itself is not so interesting.

But we will still talk about the model of Townsend. (It is part of our cultural heritage.)

- Also use Townsend's model to talk about hierarchical information structures (where the infinite regress problem can be avoided)

Forecasting the Forecasts of Others, Townsend JPE 1983

Output sector i

$$y_t^i = f_0 k_t^i$$

Market clearing price

$$\begin{aligned} P_t^i &= -b_1 Y_t^i + z_t^i \\ z_t &= \theta_t + \epsilon_t^i \end{aligned}$$

where

$$\theta_t = \rho\theta_{t-1} + v_t$$

and $\epsilon_t^i \sim N(0, \sigma_\epsilon^2)$ and $v_t \sim N(0, \sigma_v^2)$

Forecasting the Forecasts of Others, Townsend JPE 1983

The firms profit max problem:

$$\max_{\{k_t^i\}_{t=1}^{\infty}} E_0^i \sum_{t=0}^{\infty} \beta^t \left[P_t^i f_0 k_t^i - \frac{f_1}{2} \left(k_t^i\right)^2 - \frac{f_2}{2} \left(k_{t+1}^i - k_t^i\right)^2 \right]$$

$$f_0, f_2 \quad > \quad 0, \qquad f_1 \geq 0$$

Decision rule

$$k_{t+1} = \lambda_1 k_t^i + \frac{f_0 \beta \lambda_1}{f_2} \sum_{j=0}^{\infty} (\beta \lambda_1)^j E\left(P_{t+1+j}^i \mid \Omega_t^i\right)$$

gives a law of motion for aggregate industry i capital stock

$$K_{t+1}^i = h_1 K_t^1 + h_2 M_t^i$$

where

$$M_t^i = E\left(\theta_t \mid \Omega_t^i\right)$$

Hierarchical information structure

- Industry 1 is "self contained":
  - Does not observe prices in any other industry
- Industry 2 observes prices in Industry 1, but there is no trade or other "real" interaction across sectors
  - Industry 2 uses observation of price in Industry 1 to form an estimate of θ_t

The only link between industries is that they both try to estimate the same unobservable state

Industry 1 filtering problem

By Kalman filtering

$$M_t^1 = \rho M_{t-1}^1 + G_t \left(z_t^1 - \rho M_{t-1}^1 \right)$$

or with Townsend's notation:

$$M_t^1 = \alpha_0 M_{t-1}^1 + \alpha_1 z_t^1$$

Industry 1 law of motion

$$\begin{bmatrix} K^1_{t+1} \\ M^1_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & 0 \\ 0 & \alpha_0 & \alpha_1\rho \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_1 v_{t+1} + \alpha_1 \epsilon^1_{t+1} \\ v_{t+1} \end{bmatrix}$$

Given the decision rule, industry 1 is solved.

Industry 2 filtering problem

Firms in Industry 2 also want to estimate the current value of θ_t and they observeas well as the history of prices in industry 1 so Industry 2's information set is

$$\Omega_t^2 = \left\{ z_s^2, P_s^1, P_s^2, M_s^2, K_s^2 : s = 0, 1, 2, ..., t \right\}$$

Industry 2 filtering problem

Standard form for the Kalman filter:
State equation

$$\begin{bmatrix} K^1_{t+1} \\ M^1_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & 0 \\ 0 & \alpha_0 & \alpha_1\rho \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_1 v_{t+1} + \alpha_1 \epsilon^1_{t+1} \\ v_{t+1} \end{bmatrix}$$

Measurement equation

$$\begin{bmatrix} z^2_t \\ P^1_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -b_1 f_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} \epsilon^2_t \\ \epsilon^1_t \end{bmatrix}$$

Industry 2 filtering problem

We can then write the complete system as

$$\begin{bmatrix} K^1_{t+1} \\ M^1_{t+1} \\ \theta_{t+1} \\ K^2_{t+1} \\ M^2_{t+1} \\ E\left[K^1_{t+1} \mid \Omega^2_{t+1}\right] \\ E\left[M^1_{t+1} \mid \Omega^2_{t+1}\right] \end{bmatrix} = A \begin{bmatrix} K^1_t \\ M^1_t \\ \theta_t \\ K^2_t \\ M^2_t \\ E\left[K^1_t \mid \Omega^2_t\right] \\ E\left[M^1_t \mid \Omega^2_t\right] \end{bmatrix} + C \begin{bmatrix} v_t \\ \epsilon^1_t \\ \epsilon^2_t \end{bmatrix}$$

The mechanics of a hierarchical information structures

What breaks the infinite regress of expectations?

- Nested information sets and property of projections
    - Information set of Industry 1 nested in that of Industry 2 so that $\Omega_t^1 \subseteq \Omega_t^2$
    - We know from the properties of projections that $\mathcal{P}_1\mathcal{P}_2X = \mathcal{P}_1X$ and $\mathcal{P}_2\mathcal{P}_1\mathcal{P}_2X = \mathcal{P}_1X$ if $\Omega^1 \subseteq \Omega^2$

Intuition: You cannot predict the error of a strictly better informed agent so $\mathcal{P}_1\left(X - \mathcal{P}_2X\right) = 0 \implies \mathcal{P}_1X = \mathcal{P}_1\mathcal{P}_2X$

So what if firms in industry 1 also can observe prices in Industry 2?

Then we need to include industry 1's expectations about industry 2 expectations about .....industry 1's expectations of θ_t... and so on

- This is the infinite regress of expectations problem: Natural state representations tend to become infinite.

Townsend suggested that we assume

- that shocks are observed perfectly with a finite lag and
- use projection methods

This works since projections on a finite dimensional observations vector now spans the state and we can then compute

$$E[\theta_t | z_t^i, z_{t-1}^i, P_t^j, P_{t-1}^j, \theta_{t-2}]$$

Two reasons Townsend's model is special

Bugs or features?

- No real strategic interaction: Markets are only linked informationally
- In fact, there is no private information in equilibrium as shown by Sargent (JEDC 1991)

You'll be the judge.

Dynamic Higher Order Expectations Nimark (2010)

Solving dynamic models with private information

- Recursive formulation with an explicit role for higher order expectations

Impose common knowledge of rationality

- By it self does not solve the "infinite regress problem" but makes thinking about higher order expectations tractable
- Show that of expectations diminishes "fast enough" as order increases which allows for an arbitrarily good approximation
- Use Singleton's (1987) model of asset pricing with disparately informed traders as vehicle for the argument

The Singleton (1987) asset pricing model

Trader j's demand

$$z_t^d(j) = \frac{\left(E\left[p_{t+1} \mid I_t(j)\right] - \left(1+\bar{r}\right) p_t\right)}{\gamma\delta}$$

Supply

$$\begin{aligned} z_t^s &= \xi p_t + \theta_t + \epsilon_t \\ \theta_t &= \rho\theta_{t-1} + v_t \end{aligned}$$

The Singleton asset pricing model

Price Euler equation

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid I_t(j)\right] \quad dj \right) - \delta\gamma\lambda\left[\theta_t + \epsilon_t\right]$$

where

$$0 < \lambda < 1$$

Trader j's information set

$$\begin{aligned} I_t(j) &= \{s_{t-T}(j), p_{t-T} : T \geq 0\} \\ s_t(j) &= \theta_t + \eta_t(j) \end{aligned}$$

Singleton:

$$I_t^S = \{s_{t-T}(j), p_{t-T} : T \geq 0; v_{t-T}, \epsilon_{t-T} : T \geq 2\}$$

Singleton's Solution method

Same as Townsend's:

- Assume that everything is revealed with a two period lag:

$$\begin{aligned} p_t &= \lambda\left(\int E\left[p_{t+1} \mid s_t(j), s_{t-1}(j), p_t, p_{t-1}, \theta_{t-2}\right] \ dj\right) \\ &\quad -\delta\gamma\lambda\left[\theta_t + \epsilon_t\right] \end{aligned}$$

This allows for a finite and non-expanding dimension of state.

We will show how to solve the model without making this assumption.

Notation

Agents are indexed by $j \in (0,1)$

$$\begin{aligned}
\theta^{(0)}_{t|t} &\equiv \theta_t \\
\theta^{(1)}_{t|t} &\equiv \int E\left[\theta^{(0)}_{t|t} \mid I_t(j)\right] \quad dj \\
\theta^{(2)}_{t|t} &\equiv \int E\left[\theta^{(1)}_{t|t} \mid I_t(j)\right] \quad dj \\
\theta^{(k)}_{t|t} &\equiv \int E\left[\theta^{(k-1)}_{t|t} \mid I_t(j)\right] \quad dj
\end{aligned}$$

Dynamic Notation

$$
\begin{aligned}
\theta^{(1)}_{t+1|t} &\equiv \int E\left[\theta_{t+1} \mid \Omega_t(j)\right] \quad dj \\
\theta^{(2)}_{t+2|t+1|t} &\equiv \int E\left[\theta^{(1)}_{t+2|t+1} \mid \Omega_t(j)\right] \quad dj \\
\theta^{(k)}_{t+k|\cdots|t} &\equiv \int E\left[\theta^{(k-1)}_{t+k|\cdots|t+1} \mid \Omega_t(j)\right] \; dj
\end{aligned}
$$

Notation cont.

Denote a vector consisting of a *hierarchy of expectations* (from order zero to k)

$$\theta_{t|t}^{(0:k)} = \begin{bmatrix} \theta_{t|t}^{(0)} \\ \theta_{t|t}^{(1)} \\ \vdots \\ \theta_{t|t}^{(k)} \end{bmatrix}$$

Constructing a law of motion for expectations using common knowledge of rationality

- Illustrate how common knowledge of rationality impose structure on higher order expectations
- A simple example (no economics yet)

Estimating an unobservable process

The true process is an AR(1)

$$\theta_t = \rho\theta_{t-1} + v_t$$

In period t agent j observes the private noisy signal $s_t(j)$

$$\begin{aligned} s_t(j) &= \theta_t + \eta_t(j), \\ \eta_t(j) &\sim N\left(0, \sigma_\eta^2\right) \ \forall \ j \end{aligned}$$

Updating equation

$$\theta_{t|t}^{(1)}(j) = (1 - g_1)\,\rho\theta_{t-1|t-1}^{(1)}(j) + g_1 s_t(j)$$

Higher order estimates

A new state space system

$$\begin{bmatrix} \theta_t \\ \theta^{(1)}_{t|t} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ g_1\rho & (1-g_1)\rho \end{bmatrix} \begin{bmatrix} \theta_{t-1} \\ \theta^{(1)}_{t-1|t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ g_1 \end{bmatrix} v_t$$

$$s_t(j) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_t \\ \theta^{(1)}_{t|t} \end{bmatrix} + \eta_t(j)$$

Higher order estimates

Agent j can now form expectations of actual and average first order expectations using the new updating equation

$$\begin{bmatrix} \theta^{(1)}_{t|t}(j) \\ \theta^{(2)}_{t|t}(j) \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ g_1\rho & (1-g_1)\rho \end{bmatrix} \begin{bmatrix} \theta^{(1)}_{t-1|t-1}(j) \\ \theta^{(2)}_{t-1|t-1}(j) \end{bmatrix}$$

$$+ \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}' \left(\begin{bmatrix} \theta_t \\ \theta^{(1)}_{t|t} \end{bmatrix} - \begin{bmatrix} \theta^{(1)}_{t-1|t-1}(j) \\ \theta^{(2)}_{t-1|t-1}(j) \end{bmatrix} \right) + \eta_t(j) \right)$$

Higher order estimates

Again taking averages gives a law of motion for $\theta_{t|t}^{(0:2)}$

$$\begin{bmatrix} \theta_{t|t}^{(0)} \\ \theta_{t|t}^{(1)} \\ \theta_{t|t}^{(2)} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ g_1\rho & (1-g_1)\,\rho & 0 \\ g_2\rho & (g_1-g_2)\rho & (1-g_1)\,\rho \end{bmatrix} \begin{bmatrix} \theta_{t-1|t-1}^{(0)} \\ \theta_{t-1|t-1}^{(1)} \\ \theta_{t-1|t-1}^{(2)} \end{bmatrix} + \begin{bmatrix} 1 \\ g_1 \\ g_2 \end{bmatrix} v_t$$

Bounded variance of higher order expectations

Lemma: The variance of trader j's expectation of θ_t is bounded by the variance of θ_t, i.e.

$$E\left[\theta_t\right]^2 \geq E\left[\theta_t^{(1)}(j)\right]^2$$

Proof: Define trader j's first order expectation error $\varepsilon_t^{(1)}(j)$ as

$$\theta_t \equiv \theta_t^{(1)}(j) + \varepsilon_t^{(1)}(j)$$

The error $\varepsilon_t^{(1)}(j)$ is orthogonal to $\theta_t^{(1)}(j) \in \Omega_t(j)$ so we have

$$E\left[\theta_t\right]^2 = E\left[\theta_t^{(1)}(j)\right]^2 + E\left[\varepsilon_t^{(1)}(j)\right]^2$$

The proof then follows from the fact that variances are non-negative so that

$$E\left[\theta_t\right]^2 \geq E\left[\theta_t^{(1)}(j)\right]^2$$

Bounded variance of higher order expectations

Lemma: The variance of the average expectation of θ_t is bounded by the variance of θ_t, i.e.

$$E\left[\theta_t\right]^2 \geq E\left[\theta_t^{(1)}\right]^2 \tag{1}$$

Proof: Trader j's first order expectations have an MA representation with variance

$$E\left[\theta_t^{(1)}(j)\right]^2 = E\left[A(L)v_t\right]^2 + E\left[B(L)\epsilon_t\right]^2 + E\left[C(L)\eta_t(j)\right]^2 \tag{2}$$

Since $\int \eta_t(j)dj = 0 \forall t$ the average first order expectation is simply

$$\begin{aligned} \theta_t^{(1)} &= A(L)v_t + B(L)\epsilon_t + \int C(L)\eta_t(j)dj && (3) \\ &= A(L)v_t + B(L)\epsilon_t && (4) \end{aligned}$$

with variance

$$E\left[\theta_t^{(1)}\right]^2 = E\left[A(L)v_t\right]^2 + E\left[B(L)\epsilon_t\right]^2 < E\left[\theta_t^{(1)}(j)\right]^2 \tag{5}$$

Bounded variance of higher order expectations

Proposition: The variance of higher order expectations of θ_t are bounded by the variance of lower order expectations, i.e.

$$E\left[\theta_t^{(k)}\right]^2 \geq E\left[\theta_t^{(k+1)}\right]^2$$

Proof: To prove the proposition, replace the definition of trader j's first order expectations error $\varepsilon_t^{(1)}(j)$ in the proof of Lemma 1 with the definition of the k order expectation error

$$\theta_t^{(k-1)} - \theta_t^{(k)}(j) \equiv \varepsilon_t^{(k)}(j)$$

Noting that the k order error $\varepsilon_t^{(k)}(j)$ is orthogonal to $\theta_t^{(k)}(j) \in \Omega_t(j)$ allows for recursively establishing the proposition for $k = 2, 3, \ldots$by following the same steps as in the proofs of Lemma 1 and 2.

Bounded variance of higher order expectations

Proposition: The variance of higher order expectations of future expectations of θ_t are bounded by the variance of lower order expectations, i.e.

$$E\left[\theta^{(k-1)}_{t+k|\cdots|t}\right]^2 \geq E\left[\theta^{(k)}_{t+k+1|\cdots|t}\right]^2$$

Proof: To prove the proposition, replace the definition of trader j's first order expectations error $\varepsilon^{(1)}_t(j)$ in the proof of Lemma 1 with the definition of the k order future expectation error

$$\theta^{(k-1)}_{t+k|\cdots|t} - \theta^{(k)}_{t+k+1|\cdots|t}(j) \equiv \varepsilon^{(k+1)}_{t+k+1|\cdots|t}(j)$$

Again, since the k order error $\varepsilon^{(k+1)}_{t+k+1|\cdots|t}(j)$ is orthogonal to $\theta^{(k)}_{t+k+1|\cdots|t}(j) \in \Omega_t(j)$, the same recursive procedure as in Proposition 1 can be applied to establish the desired result for $k = 1, 2, 3, \ldots$

Back to Singleton's asset pricing model

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid I_t(j)\right] \quad dj \right) - \delta\gamma\lambda\left[\theta_t + \epsilon_t\right]$$

where

$$0 < \lambda < 1$$

The Full Information Equilibrium Price

Iterate price Euler-equation forward

$$\begin{aligned} p_t &= -\delta\gamma\lambda\left(\theta_t + \epsilon_t\right) + \\ &\quad -(\delta\gamma\lambda)\lambda\rho\theta_t + \ldots \\ &\quad \ldots - \left(\delta\gamma\lambda\right)\left(\lambda\rho\right)^{\infty}\theta_{t+\infty} \end{aligned}$$

to get

$$p_t = -\frac{\delta\gamma\lambda}{1-\lambda\rho}\theta_t - \delta\gamma\lambda\epsilon_t$$

The Private Information Equilibrium Price: Complications

Iterate price Euler-equation forward

$$p_t = \lambda \left(\int E\left[p_{t+1} \mid I_t(j)\right] \quad dj \right) - \delta\gamma\lambda \left[\theta_t + \epsilon_t\right]$$

to get price as a function of higher order expectations of future θ

$$\begin{aligned} p_t &= -\delta\gamma\lambda\left(\theta_t + \epsilon_t\right) - (\delta\gamma\lambda)\lambda \int E\left[\theta_{t+1} \mid I_t(j)\right] dj \\ &\quad +(\delta\gamma\lambda)\lambda^2 \int E\left[\int E\left[p_{t+2} \mid I_{t+1}(j)\right] dj \mid I_t(j)\right] dj] \end{aligned}$$

or

$$p_t = -\delta\gamma\lambda\epsilon_t - \delta\gamma\lambda \sum_{k=0}^{\infty} \lambda^k \theta^{(k)}_{t+k|\cdots|t}$$

An average expectations operator

Conjecture a law of motion for hierarchy

$$\theta_{t|t}^{(0:\infty)} = M\theta_{t-1|t-1}^{(0:\infty)} + N \begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix}$$

Define new operator $H : \mathbb{R}^\infty \to \mathbb{R}^\infty$

$$H \equiv \begin{bmatrix} \mathbf{0}_{\infty \times 1} & I \end{bmatrix}$$

then higher order expectations of future fundamental are given by

$$\begin{aligned} \int E\left[\theta_{t+1} \mid I_t(j)\right] dj &= e_1' MH\theta_{t|t}^{(0:\infty)} \\ \int E\left[\int E\left[\theta_{t+2} \mid I_{t+1}(j)\right] dj \mid I_t(j)\right] dj] &= e_1'(MH)^2\theta_{t|t}^{(0:\infty)} \end{aligned}$$

...and so on.

The price function

The price of the asset can then be written as a function of the state

$$p_t = \left[\begin{array}{cccc} a_0 & a_1 & \cdots & a_\infty \end{array} \right] \left[\begin{array}{c} \theta^{(0)}_{t|t} \\ \theta^{(1)}_{t|t} \\ \vdots \\ \theta^{(\infty)}_{t|t} \end{array} \right] - \delta\gamma\lambda\epsilon_t$$

where the vector **a** resembles a discounted geometric sum of expected future fundamentals

$$\mathbf{a} \equiv \left[\begin{array}{cccc} a_0 & a_1 & \cdots & a_\infty \end{array} \right]$$

A finite dimensional approximation

Consider only the first $\overline{k}$ orders of expectations

$$p_{\overline{k},t} = -\delta\gamma\lambda\epsilon_t - \delta\gamma\lambda \sum_{k=0}^{\overline{k}} \lambda^k \theta^{(k)}_{t+k|\cdots|t}$$

We then have that

$$p_{\overline{k},t} = -\delta\gamma\lambda\epsilon_t - \delta\gamma\lambda \sum_{k=0}^{\infty} e_1' \left(\lambda M H\right)^k \theta_t^{(0:\overline{k})}$$

Define approximation error as

$$\Delta_{\overline{k}} \equiv p_t - p_{\overline{k},t}$$

A finite state representation

The variance of the price p_t is finite
Proof: We want to show that $E\left(p_t\right)^2 < \infty$. Taking variances of both sides of the expression for the equilibrium price we get

$$
\begin{aligned}
E\left(p_t\right)^2 &= (\delta\gamma\lambda)^2 \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \lambda^{(i+j)} cov\left[\theta_{t+i|\cdots|t}, \theta_{t+j|\cdots|t}\right] \\
&\quad +2\delta\gamma\lambda \sum_{j=0}^{\infty} \lambda^{j} cov\left[\theta_{t+i|\cdots|t}, \epsilon_t\right] \\
&\quad +(\delta\gamma\lambda)^2 \sigma_\epsilon^2
\end{aligned}
$$

We know from above that the covariances on right hand side are bounded and $0 < \lambda < 1$ we know that right hand side converges.

A finite state representation

Substitute in solution as function of current state state

$$p_t = \mathbf{a}\theta_{t|t}^{(0:\infty)} - \delta\gamma\lambda\epsilon_t$$

and take variances

$$\begin{aligned} E\left[p_t\right]^2 &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_i a_j cov\left[\theta_t^{(i)}, \theta_t^{(j)}\right] \\ &\quad +2\delta\gamma\lambda\sum_{j=0}^{\infty} a_j cov\left[\theta_t^{(j)}, \epsilon_t\right] \\ &\quad +\left(\delta\gamma\lambda\right)^2\sigma_\epsilon^2 \end{aligned}$$

Again, since we know that the price have finite variance, infinite sums on on right hand side must converge.

A finite state representation

The variance of the approximation error

$$
\begin{aligned}
E\left(\Delta_{\overline{k}}\right)^2 &= \sum_{j=\overline{k}+1}^{\infty} \sum_{i=\overline{k}+1}^{\infty} a_i a_j cov\left[\theta_t^{(i)}, \theta_t^{(j)}\right] \\
&\quad +2\delta\gamma\lambda \sum_{j=\overline{k}+1}^{\infty} a_j cov\left[\theta_t^{(j)}, \epsilon_t\right]
\end{aligned}
$$

must then tend to zero for the expression on previous slide to converge.

The law of motion of the expectations hierarchy

We want the form

$$\theta_{t|t}^{(0:\infty)} = M\theta_{t-1|t-1}^{(0:\infty)} + N\begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix}$$

Process for actual state

$$\theta_t = \rho\theta_{t-1} + v_t$$

Trader j's hierarchy updating equation

$$\theta_{t|t}^{(1:\infty)}(j) = M\theta_{t-1|t-1}^{(1:\infty)}(j) + K\left(S_t(j) - LM\theta_{t-1|t-1}^{(1:\infty)}(j) - Qc_t\right)$$

The law of motion of the expectations hierarchy

Signal vector is a function of the state

$$\begin{aligned} \int S_t(j) \;\; dj &= L\theta_{t|t}^{(0:\infty)} + Qc_t \\ L &= \begin{bmatrix} e_1' \\ \mathbf{a} \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 \\ \frac{\lambda\psi}{1-\lambda\psi} \end{bmatrix} \end{aligned}$$

We can write the average hierarchy updating equation as

$$\begin{aligned} \theta_{t|t}^{(1:\infty)} &= (I - KL)\, M\theta_{t-1|t-1}^{(1:\infty)} + KLM\theta_{t-1|t-1}^{(0:\infty)} \\ &\quad + KLN \begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix} + K \begin{bmatrix} 0 \\ -\delta\gamma\lambda\epsilon_t \end{bmatrix} \end{aligned}$$

The law of motion of the expectations hierarchy

Actual process and average hierarchy updating equation imply

$$
\begin{aligned}
M &= \begin{bmatrix} \rho & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ KLM_{-} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & (I - KL)\, M_{-} \end{bmatrix} \\
N &= \begin{bmatrix} e_1' \\ KLN_{-} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbf{0} & -K_2 \delta \gamma \lambda_{-} \end{bmatrix}
\end{aligned}
$$

To solve the model, find a fixed point for M, N, a and δ

The solved model

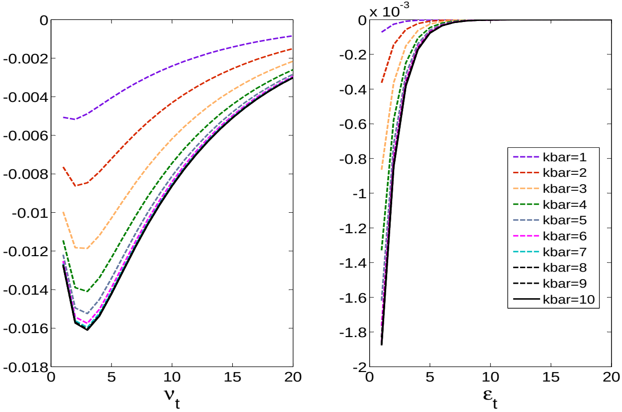
The solved model is in the form

$$\theta_{t|t}^{(0:\overline{k})} = M\theta_{t-1|t-1}^{(0:\overline{k})} + N \begin{bmatrix} v_t \\ \epsilon_t \end{bmatrix}$$

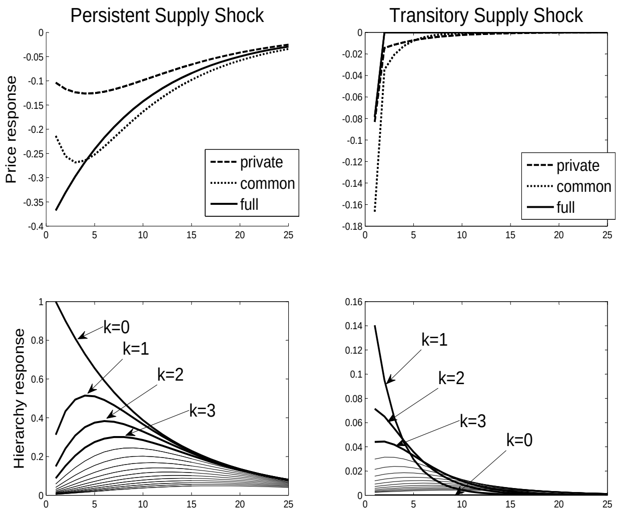
$$p_{\overline{k},t} = -\delta\gamma\lambda\epsilon_t - \delta\gamma\lambda e_1'(I - \lambda MH)^{-1}\theta_t^{(0:\overline{k})}$$

But we still need to choose $\overline{k}$

Choosing $\bar{k}$: Convergence in practice



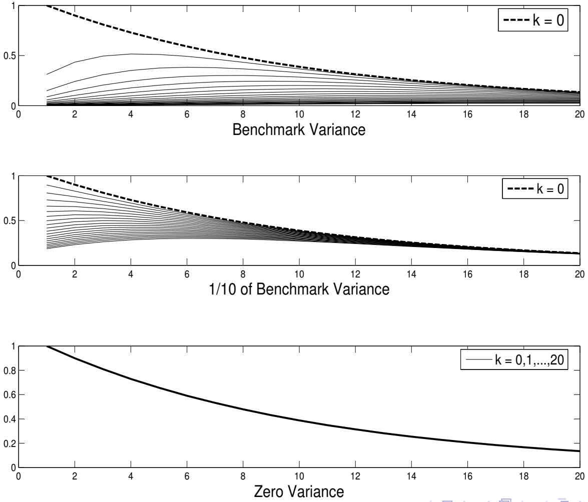
Price and hierarchy dynamics



The information revealed by prices

- Private information is not preserved in Townsend's (1983) model when equilibrium prices are observed: Sargent (1991), Kasa (2000) Pearlman and Sargent (2005)
- Walker (2007) makes similar claim about Singleton's model. But:
  - Does not prove this in Singleton's original set up
  - Makes additional assumption that the supply shock ϵ_t is directly observable
  - The price then reveals θ_t perfectly

IRF of hierarchy of θ_t and the variance of ϵ_t



The method in 3 steps

1. Impose structure on higher order expectations through common knowledge of rational expectations
2. Variance of expectations non-increasing with order of expectation
3. Impact of expectations decreasing with order of expectation

The method seems quite useful

- No need to assume lagged shock revelation
- Can handle dynamic choices in infinite horizon models
- Multi dimensional hidden state
  - E.g. Nimark (JME 2008) on pricing decisions in a general equilibrium macro model
- Endogenous hidden state
  - E.g. capital stock in RBC model with incomplete markets and idiosyncratic technology shocks (Graham and Wright 2007)
- Solution fast enough for empirical work (Melosi 2010, Nimark 2010)

That's it for today.