SPECULATION AND THE TERM STRUCTURE OF INTEREST RATES

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Abstract

A tractable equilibrium term structure model populated with rational but heterogeneously informed traders is developed and estimated. Traders take on speculative positions to exploit what they perceive to be inaccurate market expectations about future bond prices. Yield dynamics due to speculation are (i) statistically distinct from classical term structure components due to risk premia and expectations about future short rates and must be orthogonal to public information available to traders in real time and (ii) quantitatively important, potentially accounting for a substantial fraction of the variation of long maturity US bond yields.

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1. Introduction

A fundamental question in finance is what the economic forces are that explain variation in asset prices and returns. This paper demonstrates that allowing for heterogeneous information sets among rational traders introduces a speculative component in bond yields that is absent in models in which all traders share the same information. The speculative term is statistically distinct from both risk premia and terms reflecting expectations about future risk free short rates and potentially empirically important.

Many bonds, and US treasury bonds in particular, are traded in very liquid secondary markets. In such a market, the price an individual trader will pay for a long maturity bond depends on how much he thinks other traders will pay for the same bond in the future. If traders have access to different information, this price may differ from what an individual trader would be willing to pay for the bond if he had to hold it until maturity. The possibility of reselling a bond then changes its equilibrium price as traders take speculative positions in order to exploit what they perceive to be market misperceptions about future bond prices.

This paper presents and structurally estimates an equilibrium model of the term structure of interest rates that is populated with traders that engage in this type of speculative behavior. In the model, individual traders can identify bonds that, conditional on their own information sets, have a positive expected excess return. In the absence of arbitrage, expected returns in excess of the risk free rate must be compensation for risk. Traders will hold more of the bonds with a higher expected return in their portfolios. In equilibrium, the increased riskiness of a less balanced portfolio is exactly off-set by the higher expected return. We show formally that heterogeneous information introduces a source of time varying expected excess returns that, unlike the excess returns documented by for instance Fama and Bliss (1987) and Campbell and Shiller (1991), cannot be predicted conditional on past bond yields.

When aggregated, the speculative behaviour of individual traders introduces new dynamics to bond prices. We demonstrate that when traders have heterogeneous information sets, bond yields are partly determined by a speculative component that reflects traders' expectations about the error in the average, or market, expectations of future risk-free interest rates. Since it is not possible for individual traders to predict the errors that other traders make based on information available to everybody, the speculative component in bond prices must be orthogonal to publicly available information. Heterogeneous information thus introduces a third term in bond yields that is statistically distinct from the classical components of yield curve decompositions, i.e. terms due to risk premia and terms reflecting expectations about future risk-free short rates.

Despite the fact that the speculative component in bond yields must be orthogonal to public information, it is possible to quantify its importance using only publicly available data on bond yields. This is so because we as econometricians have access to the full sample of data and the speculative term is orthogonal only to public information available to traders in real time. That is, we can use public information available in period $t+1, t+2, ...$ and so on, to back out an estimate of the speculative term in period t. The estimated model suggests that speculative dynamics are quantitatively important and can explain a substantial fraction of the variation in US bond yields.

A necessary condition for traders to have any relevant private information about future bond yields is that bond prices do not perfectly reveal the state of the economy. Recent statistical evidence supports this view. In a few closely related papers, Joslin, Priebsch and Singleton (2014) and Duffee (2011) present evidence suggesting that the factors that can be found by inverting yields are not sufficient to optimally predict future bond returns. They find that while the usual level, slope and curvature factors explain virtually all of the cross sectional variation in yields, additional factors are needed to forecast excess returns. Ludvigson and Ng (2009) provide more evidence that current bond yields are not sufficient to optimally forecast bond returns. They show that compared to using only yield data, drawing on a very large panel of macroeconomic data helps predict excess returns. Stated another way, these statistical models all suggest that linear combinations of current bond yields are not sufficient to predict future bond yields optimally.

In addition to the empirical evidence cited above, we also have a priori reasons to believe that bond prices should not reveal all information relevant to predicting future bond returns. Grossman and Stiglitz (1980) argued that if it is costly to gather information and prices are observed costlessly, prices cannot fully reveal all information relevant for predicting future returns. For the bond market, the most important variable to forecast is the short interest rate. In most developed countries, the short interest rate is set by a central bank that responds to macroeconomic developments. If it is costly to gather information about the macro economy, Grossman and Stiglitz's argument implies that bond prices cannot reveal all information relevant to predict bond returns.

In practice, there is a vast amount of financial and macro economic data available that could in principle help traders to predict future bond yields. If prices do not reveal all the information that is relevant for predicting bond returns it becomes more probable that different traders will use different subsets of the available information. Here, we model this by endowing traders with partly private information that they can exploit when trading. This set up also accords well with the casual observation that at least one motive for trade in assets is possession of information that is not, or at least is not believed to be, already reflected in prices. Formally, our set-up is similar to the information structure in Diamond and Verrecchia (1981), Admati (1985), Singleton (1987), Allen, Morris and Shin (2006) and Bacchetta and van Wincoop (2006).

One implication of heterogeneous information sets is that different traders have different expectations about future bond yields. This provides us with another way of gauging the plausibility of this assumption. While bond traders' expectations are unobservable, the average cross-sectional dispersion of responses of one-quarter ahead Federal Funds Rate in the *Survey of Professional Forecasters* is about 40 basis points in the 1980-2014 sample.

There exists a very large theoretical literature that studies asset pricing with heterogeneously informed agents. Hellwig (1980), Diamond and Verecchia (1981), Admati (1985) and Singleton (1987) are some of the early references. More recent examples include papers by Allen, Morris and Shin (2006), Kasa, Walker and Whiteman (2014), Bacchetta and van Wincoop (2006, 2007), Cespa and Vives (2012) and Makarov and Rytchkov (2012). These papers either present purely theoretical models or models calibrated to explain some feature of the data. The model presented here is estimated directly using likelihood based methods.

To the best of my knowledge, this is the first paper to empirically quantify the importance of heterogeneous information sets for asset prices and returns.

2. A Bond Pricing Model

This section presents an equilibrium bond pricing model. Traders are risk averse, rational, and ex ante identical but may observe different signals relevant for predicting future bond prices. They choose a portfolio of risky bonds in order to maximize next-period wealth. Traders that have observed signals that make them more optimistic about the return of a given bond will hold relatively more of that bond in their portfolio and in equilibrium, the increased riskiness of a less balanced portfolio is exactly offset by a higher expected return. The equilibrium price of a bond is a function of the average expectations of the price of the same bond in the next period, discounted by the risk-free short interest rate. Bond prices are also affected by supply shocks that prevents equilibrium prices from revealing the average expectation of future bond prices.

The model is relatively tractable and in the section following this one it will be used to draw out the consequences for term structure dynamics of relaxing the assumption that traders all have access to the same information.

2.1. **Demand for long maturity bonds.** Time is discrete and indexed by t. As in Allen, Morris and Shin (2006) there are overlapping generations of agents who each live for two periods. Each generation consists of a continuum of households with unit mass. Each household is endowed with one unit of wealth that it invests when young. When old, households unwind their asset positions and use the proceeds to consume. Unlike in the model of Allen *et al*, the owners of wealth, i.e. the households, do not trade assets themselves. Instead, a continuum of traders, indexed by $j \in (0,1)$, trade on behalf of the households, with households diversifying their funds across the continuum of traders. While not modeled explicitly here, this set up can be motivated as a perfectly competitive limit case of the mutual funds model of Garcia and Vanden (2009) that allows uninformed households to benefit from mutual funds private information, while diversifying away idiosyncratic risk associated with individual funds. More importantly, the assumption that the ownership of the assets is separated from the privately informed traders keeps the model tractable by abstracting from information induced wealth heterogeneity.[1]

The formal structure of the model is as follows. Trader j invests one unit of wealth in period t on behalf of households born in period t. In period $t+1$ trader j unwinds the position of the now old generation of households who then use the proceeds to consume. Traders are infinitely lived and perform the same service for the next generation of households.

There are two types of assets: a risk-free one period bond with (log) return r_t and risky zero-coupon bonds of maturities $2, 3, ..., \overline{n}$ periods. Trader j chooses a vector of portfolio weights $\boldsymbol{\alpha}_t^j$ in order to maximize the expected log of wealth under management W_{t+1}^j in period $t+1$. That is, trader j solves the problem

$$\max_{\boldsymbol{\alpha}_t^j} E\left[\log W_{t+1}^j \mid \Omega_t^j\right] \tag{2.1}$$

[1]Xiong and Yan (2010) present a calibrated difference-in-beliefs model that they use to analyze the interaction of beliefs and wealth dynamics and how that affect the term structure of interest rates.

subject to

$$W_{t+1}^{j} = 1 + r_{t,j}^{p} \tag{2.2}$$

where Ω_t^j denotes trader j's information set and $r_{t,j}^p$ is the log return of the portfolio chosen by trader j in period t. All traders observe the short risk-free rate r_t as well as the price of all bonds.

In equilibrium, log returns of individual bonds will be normally distributed. However, the log return on a portfolio of assets with individual log normal returns is not normally distributed. Following Campbell and Viceira (2002a, 2002b) we therefore use a second order Taylor expansion to approximate the log excess portfolio return as

$$r_{t,j}^{p} - r_t = \boldsymbol{\alpha}_t^{\prime j} \mathbf{rx}_{t+1} + \frac{1}{2}\boldsymbol{\alpha}_t^{\prime j} diag\left[\Sigma_{rx,t}^{j}\right] - \frac{1}{2}\boldsymbol{\alpha}_t^{\prime j}\Sigma_{rx,t}^{j}\boldsymbol{\alpha}_t^{j} \tag{2.3}$$

where $\mathbf{rx}_{t+1}$ is a vector of period $t+1$ excess returns on bonds defined as

$$\mathbf{rx}_{t+1} \equiv \begin{bmatrix} p_{t+1}^{1} - p_t^{2} - r_t \\ p_{t+1}^{2} - p_t^{3} - r_t \\ \vdots \\ p_{t+1}^{\overline{n}-1} - p_t^{\overline{n}} - r_t \end{bmatrix} \tag{2.4}$$

and p_t^n is the log price of a bond with n periods to maturity. The matrix $\Sigma_{rx,t}^j$ is the covariance of log bond returns conditional on trader j's information set. In equilibrium, conditional returns will be normally distributed, time invariant and with a common conditional covariance across all traders. We can thus suppress the time subscripts and trader indices on the conditional return covariance matrix and write Σ_{rx} instead of $\Sigma_{rx,t}^j$ for all t and j. Maximizing the expected log wealth (2.2) with respect to $\boldsymbol{\alpha}_t^j$ then gives the optimal portfolio weights

$$\boldsymbol{\alpha}_t^{j} = \Sigma_{rx}^{-1} E\left[\mathbf{rx}_{t+1} \mid \Omega_t^{j}\right] + \frac{1}{2}\Sigma_{rx}^{-1} diag\left[\Sigma_{rx}\right]. \tag{2.5}$$

The higher return a trader expects to earn on a bond, the more will he hold of it in his portfolio. However, risk aversion prevents the most optimistic trader from demanding all of the available supply. Since each trader j has one unit of wealth to invest, integrating the portfolio weights (2.5) across traders yields the aggregate demand for bonds.

2.2. **Bond supply.** The vector of bond supply $\mathbf{s}_t$ is stochastic and distributed according to

$$\mathbf{s}_t = \boldsymbol{\mu} + \Sigma_{rx}^{-1}\mathbf{v}_t : \mathbf{v}_t \sim N\left(\mathbf{0}, VV'\right) \tag{2.6}$$

To simplify notation, the vector of supply shocks $\mathbf{v}_t$ are normalized by the inverse of the conditional variance of bond prices Σ_{rx}^{-1}. The supply shocks $\mathbf{v}_t$ play a similar role here as the noise traders in Admati (1985). That is, they prevent equilibrium prices from fully revealing the information held by other traders. While there may be some uncertainty about the total number of bonds outstanding, a more appealing interpretation of the supply shocks is in terms of effective supply, as argued by Easley and O'Hara (2004). They define the "float" of an asset as the actual number of assets available for trade in a given period.

2.3. **Equilibrium bond prices.** Equating aggregate demand $\int \boldsymbol{\alpha}_t^j dj$ with supply $\mathbf{s}_t$ and solving for the log price p_t^n gives

$$p_t^n = \frac{1}{2}\sigma_n^2 - r_t + \int E\left[p_{t+1}^{n-1} \mid \Omega_t^j\right] dj - \Sigma_{rx}^n \boldsymbol{\mu} - v_t^n \tag{2.7}$$

where $\frac{1}{2}\sigma_n^2$ and v_t^n are the relevant elements of $\frac{1}{2}diag\left[\Sigma_{rx}\right]$ and $\mathbf{v}_t$ respectively and Σ_{rx}^n is the n^{th} row of Σ_{rx}. The price of an $n-$period bond in period t thus depends on the average expectation in period t of the price of an $n-1$ period bond in period $t+1$.

2.4. **The term structure of interest rates and higher order expectations.** The log price of a one-period risk-free bond is the inverse of the short interest rate, i.e.

$$p_t^1 = -r_t. \tag{2.8}$$

Taking this as the starting point we can apply (2.7) recursively to find the price of long maturity bonds. The log price of a two period bond is then given by

$$p_t^2 = \frac{1}{2}\sigma_2^2 - r_t - \int E\left[r_{t+1} \mid \Omega_t^j\right] dj - \Sigma_{rx}^2 \boldsymbol{\mu} - v_t^2 \tag{2.9}$$

i.e. p_t^2 is a function of the average first order expectations about the next period risk free rate r_t.

Continuing with the same logic, the price of a three period bond is the average period t expectation of the price of a two period bond in period $t+1$, discounted by the short rate r_t. Leading (2.9) by one period and substituting into (2.7) with $n=3$ gives

$$\begin{aligned} p_t^3 \quad = \quad & \frac{1}{2}\left(\sigma_2^2 + \sigma_3^2\right) - \Sigma_p^2 \boldsymbol{\mu} - \Sigma_p^3 \boldsymbol{\mu} \\ & -r_t - \int E\left[r_{t+1} \mid \Omega_t^j\right] dj \\ & - \int E\left[\int E\left[r_{t+2} \mid \Omega_{t+1}^{j'}\right] dj' \mid \Omega_t^j\right] dj \\ & -v_t^3. \end{aligned} \tag{2.10}$$

The expression (2.10) demonstrates that the period t price of a three period bond is a function not only of the average expectation of future risk-free interest rates but also of higher order expectations. That is, the price partly depends on the average expectation in period t of the average expectation in period $t+1$ of the risk-free rate in period $t+2$. In general, second and higher order expectations do not coincide with first order expectations when traders have heterogenous information sets. The price of a 3 period bond will then deviate from the "consensus value" of the bond, i.e. the price the bond would have if it reflected only the average (first order) period t expectation about risk-free interest rates in period $t+1$ and $t+2$.

Higher order expectations will matter for the price of all bonds of maturity $n > 2$. Recursive forward substitution of (2.7) can be used to find a general expression for the price of an n

period bond as

$$p_t^n = \sum_{i=2}^{n}\left(\frac{1}{2}\sigma_i^2 - \Sigma_{rx}^i \boldsymbol{\mu}\right) - \sum_{k=0}^{n-1} r_{t:t+k}^{(k)} - v_t^n \tag{2.11}$$

where we used the more compact notation

$$r_{t:t+k}^{(k)} \equiv \int E\left[\int E\left[... \int E\left[r_{t+k} \mid \Omega_{t+k-1}^{j''}\right] dj'' ... \mid \Omega_{t+1}^{j'}\right] dj' \mid \Omega_t^j\right] dj \tag{2.12}$$

for a k order expectation of r_{t+k}. The price of an n-period bond thus depends on average expectations of future short rates of order up to $n-1$. As usual, the yield y_t^n of an n period bond can be computed as $y_t^n = -n^{-1} p_t^n$.

2.5. **Unconditional bond yields.** Traders form model consistent expectations which implies that the unconditional mean of the higher order expectations of the risk-free rate in the bond price equation (2.11) coincide with the true unconditional mean. The unconditional yield of an n period bond is thus given by

$$E\left[y_t^n\right] = E\left[r_t\right] + n^{-1} \sum_{i=2}^{n}\left(\Sigma_{rx}^i \boldsymbol{\mu} - \frac{1}{2}\sigma_i^2\right) \tag{2.13}$$

The term $E\left[r_t\right]$ in (2.13) reflects how the average risk-free short rate affects long maturity yields. The second term, $n^{-1}\sum_{i=2}^{n} \Sigma_{rx}^i \boldsymbol{\mu}$, captures both risk-premia via the covariances in Σ_{rx}^i and supply effects from the vector $\boldsymbol{\mu}$. Risk premia will be high if the conditional variances are large or if conditional excess returns are positively correlated. The average supply of bonds may increase or decrease bond yields depending on whether the conditional returns are positively or negatively correlated. The last component on the right hand side of (2.13) is a Jensen's inequality term due to the log transformation.

Unconditional yields depend on the conditional covariance of bond returns and will thus be influenced by traders' information sets. However, the unconditional yields are known to the traders in the model and do not influence their filtering problem.

3. Heterogeneous information, excess returns and speculation

In this section we derive the main theoretical implications of relaxing the assumption that all traders have access to the same information. First, we will demonstrate that heterogeneous information introduces trader-specific risk premia. We prove formally that, unlike classical bond risk premia, risk premia due to information heterogeneity must be orthogonal to publicly available information. Second, we define the speculative portfolio as the component of a trader's portfolio held in order to exploit what he perceives to be inaccurate market expectations about next period bond prices. Third, we derive the speculative component of bond yields and prove that, just like the trader specific component in risk-premia, it must be orthogonal to publicly available information.

3.1. **Heterogenous information and expected excess returns.** The holding period return on a zero-coupon bond depends on how its price changes over time. To the extent that different traders have different expectations about future bond prices, they will also have different expectations about bond returns. In our model, this can be seen most clearly from the definition of the realized excess return on an n period bond

$$rx_{t+1}^{n} \equiv p_{t+1}^{n-1} - p_t^n - r_t. \tag{3.1}$$

The excess return that trader j expects to earn on an n period bond is thus given by

$$E\left[rx_{t+1}^{n} \mid \Omega_t^j\right] = E\left[p_{t+1}^{n-1} \mid \Omega_t^j\right] - p_t^n - r_t. \tag{3.2}$$

since current bond prices and short rates are directly observed by all traders. By substituting out the current bond price p_t^n using the expression (2.7), the excess return that trader j expects to earn on an n period bond can be expressed as a sum of a trader specific and a common component

$$E\left[rx_{t+1}^{n} \mid \Omega_t^j\right] = \underbrace{E\left[p_{t+1}^{n-1} \mid \Omega_t^j\right] - \int E\left[p_{t+1}^{n-1} \mid \Omega_t^{j'}\right] dj'}_{trader\ specific} - \underbrace{\frac{1}{2}\sigma_n^2 + \Sigma_{rx}^n \boldsymbol{\mu} + v_t^n}_{common}$$

In equilibrium, a positive expected excess return can only be earned as compensation for risk. Since individual portfolios are determined by expected excess returns and because traders are risk-averse, a trader who is more optimistic than the average trader about the return of an n period bond will hold more of it in his portfolio and have a larger conditional portfolio return variance. The risk that a more optimistic trader is compensated for is thus the risk associated with holding a portfolio with a higher conditional variance of returns.

In the absence of information heterogeneity, the expected excess return would be determined by the constants $\frac{1}{2}\sigma_n^2 + \Sigma_{rx}^n \boldsymbol{\mu}$ and the supply shock v_t^n. There is thus a time-varying component in risk premia that is common to all traders. However, the component of excess return that is due to information heterogeneity is statistically distinct from the common component since it must be orthogonal to public information in real time. Before proving this statement formally, we first define the relevant information set.

Definition 1. *The public information set Ω_t at time t is the intersection of the period t information sets of all traders*

$$\Omega_t \equiv \bigcap_{j \in (0,1)} \Omega_t^j. \tag{3.3}$$

Proposition 1. *The trader specific component in the expected excess return*

$$E\left[rx_{t+1}^{n} \mid \Omega_t^j\right] - \int E\left[rx_{t+1}^{n} \mid \Omega_t^{j'}\right] dj' \tag{3.4}$$

is orthogonal to public information in real time.

Proof. For any random variable X, the law of iterated expectations (e.g. Brockwell and Davis 2006) states that

$$E\left(E\left[X \mid \Omega'\right] \mid \Omega\right) = E\left(X \mid \Omega\right) \tag{3.5}$$

if and only if $\Omega \subseteq \Omega'$. Take expectations of the left hand side of (3.4) with respect to the public information set (3.3) and use that $\Omega_t \subseteq \Omega_t^j$ to get

$$E\left[\left(E\left[rx_{t+1}^n \mid \Omega_t^j\right] - \int E\left[rx_{t+1}^n \mid \Omega_t^{j'}\right] dj'\right) \mid \Omega_t\right] = E\left[rx_{t+1}^n - rx_{t+1}^n \mid \Omega_t\right] \tag{3.6}$$

$$= 0 \tag{3.7}$$

which completes the proof. □

In two influential papers, Fama and Bliss (1987) and Campbell and Shiller (1991) argued that excess returns on bonds can be predicted using current yields. One implication of Proposition 1 is thus that the trader specific component in expected excess return is statistically distinct from the classic predictable excess returns documented in these papers.

3.2. **The speculative portfolio.** Traders that have different return expectations will hold different portfolios. We define the speculative component of trader $j's$ portfolio as the bonds trader j holds because he believes average return expectations are inaccurate. That is, the speculative component in trader j's portfolio is the difference between trader j's actual portfolio and the portfolio trader j believes the average trader holds and it is given by

$$\boldsymbol{\alpha}_t^j - E\left(\int \boldsymbol{\alpha}_t^i di \mid \Omega_t^j\right) = \Sigma_{rx}^{-1} E\left[\left(\mathbf{rx}_{t+1} - \int E\left(\mathbf{rx}_{t+1} \mid \Omega_t^i\right) di\right) \mid \Omega_t^j\right]. \tag{3.8}$$

The speculative component in trader j's portfolio is thus the (covariance weighted) difference between trader j's expected returns and the returns that trader j believes the average trader expects to earn on bonds. If all other traders shared trader j's expectations, bond prices would adjust until all traders, including trader j, would hold the average portfolio. Trader j thus owns some bonds only because he believes that the average, or market, expectations about bond returns are incorrect. Below, we will use the estimated model to quantify how the speculative portfolio of the average trader reacts to the shocks that drive bond yields.

3.3. **Speculation, bond prices and public information.** When aggregated, the speculative behaviour of individual traders affects the demand for bonds, and in extension, bond prices. Above, we defined the speculative portfolio in terms of differences in one-period return expectations which depend on the expected next period price. Of course, the next period price may will also be partly determined by speculative behavior, and expectations about the price further into the future, and so on. In order to account for the total effect of speculation on a bond's price, it is helpful to first define a useful counter-factual price.

3.3.1. *The consensus price.* Following Allen, Morris and Shin (2006) we define the *consensus price* $\overline{p}_t^n$ of an n-period bond as the price that would "reflect the 'average opinion' of the fundamental value of the asset properly discounted". The consensus price is thus the counter-factual price a bond would have, if by chance, all traders happened to share the average trader's period t expectations about the risk-free interest rates between period t and $t + n - 1$ and this fact was common knowledge. It can be found by replacing the higher order expectations of the risk-free rate in (2.11) with the average trader's first order expectations

$$\overline{p}_t^n \equiv \frac{1}{2}\sum_{i=2}^{n}\left(\sigma_i^2 - \Sigma_{rx}^i \boldsymbol{\mu}\right) - \int \sum_{k=0}^{n-1} E\left[r_{t+k} \mid \Omega_t^j\right] dj - v_t^n. \tag{3.9}$$

We use the counter-factual consensus price $\overline{p}_t^n$ to define the speculative component in actual bond prices.

3.3.2. *The speculative component in bond prices.* The speculative component in bond prices is the difference between the actual price and the counter-factual consensus price. Taking the difference between (2.11) and (3.9), we get

$$p_t^n - \overline{p}_t^n = \sum_{k=0}^{n-1}\left(\int E\left[r_{t+k} \mid \Omega_t^j\right] dj - r_{t:t+k}^{(k)}\right). \tag{3.10}$$

The speculative component in an n-period bond price can thus be expressed as the difference between first and higher order expectations about future short interest rates.[2]

It is straightforward to show that the speculative component in bond prices must be orthogonal to public information.

Proposition 2. *The speculative term $p_t^n - \overline{p}_t^n$ is orthogonal to public information in real time, i.e.*

$$E\left[p_t^n - \overline{p}_t^n \mid \Omega_t^p\right] = 0 \tag{3.11}$$

Proof. In the Appendix. □

While the formal proof of Proposition 2 is given in the Appendix, the logic is simple and intuitive. The speculative component (3.10) consists of higher order expectations errors about the risk-free interest rate, that is, predictions about other traders' prediction errors. By definition, the public information set is available to all traders. Clearly, it is not possible for an individual trader to predict the errors that other traders are making by using information that is available also to them. The speculative component in a bond's price must therefore be orthogonal to public information available in real time.

Allen, Morris and Shin (2006) argue that with privately informed traders, asset prices may display "drift", i.e. slow adjustment to shocks with several small price changes in the same direction. While this is true if one conditions on the actual value of the fundamental, Proposition 2 demonstrates that there should be no discernible drift caused by private information that can be identified simply by observing prices or other information that is publicly available in real time.

That the speculative component in bond prices must be orthogonal to public information available is a consequence of that traders form rational model consistent expectations. This also makes the speculative component derived here different from the speculative component in the difference-in-beliefs model of Xiong and Yan (2010). In their model, traders are boundedly rational and do not condition on bond prices when they form expectations about

[2] In a different context, Bacchetta and Wincoop (2006) shows that a similar term (which they label the "higher order wedge") can be expressed as an average expectation error of the innovations to the fundamental process in their model.

future bond yields. To an outside econometrician, the speculative component in that model looks like classical risk premia, i.e. it makes excess returns predictable based on current bond yields.

3.4. **Decomposing bond prices.** There exists a very large empirical term structure literature that implicitly or explicitly decomposes long-term interest rates into expectations about future risk-free short interest rates and risk-premia, e.g. Cochrane and Piazzesi (2008) and Joslin, Singleton and Zhu (2011). The premise for these type of two-way decompositions is that risk premia and expectations about future risk free interest rates are sufficient to completely account for the yield-to-maturity of a bond. However, heterogeneous information introduces a third component to bond yields due to speculative behaviour by traders.

Add and subtract the consensus price (3.9) from the right hand side of the price of an n-period bond (2.11) and rearrange to get

$$\begin{aligned} p_t^n \quad = \quad & \underbrace{\sum_{k=0}^{n-1} \int E\left[r_{t+k} \mid \Omega_t^j\right] dj}_{Average\ 1^{st}\ order\ short-rate\ expectations} \\ & + \sum_{k=0}^{n-1} \underbrace{\left(\int E\left[r_{t+k} \mid \Omega_t^j\right] dj - r_{t:t+k}^{(k)} \right)}_{Speculative\ component} \\ & \underbrace{+ \frac{1}{2} \sum_{i=2}^{n} \left(\sigma_i^2 - \Sigma_{rx}^i \boldsymbol{\mu}\right) - v_t^n}_{Common\ risk\ premia} \end{aligned} \tag{3.12}$$

The price of a long-maturity bond can thus be expressed as the sum of average first order expectations about future risk-free short rates, a speculative component due to higher order prediction errors and a risk-premia component common to all traders. From Proposition 2 we know that the speculative component must be orthogonal to public information in real time. The speculative component is thus statistically distinct from both common risk premia and first order expectations about future risk-free rates.

In a model with perfect or common information, the speculative component would be zero at all times and bond prices would then be a function only of common short rate expectations and risk premia. The speculative component would also be zero if there were no secondary markets for trading bonds. In the absence of secondary markets, bonds can only be purchased when they are issued and must then be held until maturity. In such a setting, the expectation of other traders' expectations would not matter for the equilibrium price, since the price of a zero coupon bond at maturity is simply its face value, which is known to all traders and does not depend on the expectations of other traders. The new dynamics introduced to the term structure by heterogeneous information sets are thus dependent on the fact that long maturity bonds can be traded in secondary markets.

This ends the theoretical part of the paper. Before turning to the data, we can summarize our findings so far. With heterogeneous information sets, individual traders can identify and take advantage of predictable excess returns that would be absent in a model with only common information. We also demonstrated that the new bond price dynamics introduced by speculative behavior must be orthogonal to public information. This has an interesting empirical implication: Speculative dynamics cannot be detected using public data in real time. However, as econometricians we can use public information from periods $t+s : s > 0$ to extract an estimate of the speculative component in bond yields in period t. To do so, we need to specify explicit processes for the risk-free short rate, bond supply and traders' information sets.

4. Empirical Specification

Above, bond prices were derived as functions of higher order expectations of future short rates. In order to have an operational model that we can use to quantify the implications of heterogenous information, we here specify explicit processes for the short rate, the supply of long maturity bonds and the information sets of the traders. In this section we also describe how the model can be solved and estimated.

4.1. **The short rate and the exogenous factors.** The short interest rate r_t is an affine function of a vector of exogenous factors $\mathbf{x}_t$

$$r_t = \delta_0 + \delta_x \mathbf{x}_t \tag{4.1}$$

where the factors follow the vector autoregressive process

$$\mathbf{x}_t = F\mathbf{x}_{t-1} + C\mathbf{u}_t : \mathbf{u}_t \sim N(0, I). \tag{4.2}$$

We will normalize the short rate and factor processes by assuming that δ_x is a vector of ones, F is a diagonal matrix with the i^{th} diagonal element denoted f_i and C is a lower triangular matrix with the c_{ij} in the i^{th} row and j^{th} column. Normalizing F and C to be diagonal and lower triangular do not restrict the dynamics of r_t.

In the estimated model, $\mathbf{x}_t$ is a four dimensional vector. This gives a sufficiently high dimensional latent state to make the filtering problem of traders non-trivial, while keeping the model computationally tractable.

4.2. **Parameterizing bond supply.** The bond supply distribution (2.6) is parameterized as follows. The mean supply vector $\boldsymbol{\mu}$ has a typical element μ_n given by $\mu\lambda^n$. The parameter λ governs how the average supply of bonds changes with maturity n. With $\lambda > 1$, supply increases with maturity, and conversely, $\lambda < 1$ implies that average supply decreases with maturity.

The matrix V, i.e. the square root of the covariance of the supply shocks $\mathbf{v}_t$, is diagonal with the n^{th} diagonal element given by σn^{-1} This parameterization implies that the standard deviation of the direct effect of supply shocks on bond yields is constant across maturities and that supply shocks are independent across maturities. These restrictions are imposed in order to economize on the number of free parameters but preliminary estimates suggests that they imply very small costs in terms of fit.

4.3. **Traders' information sets.** All traders observe a vector of public signals containing the current short rate r_t and bond yields of maturity 2,3,...,$\overline{n}$ collected in the vector $\mathbf{y}_t$. Heterogeneous information is introduced through trader-specific signals about the latent factors $\mathbf{x}_t$. The vector of private signals $\mathbf{z}_t^j$ observed by trader j is specified as

$$\mathbf{z}_t^j = \mathbf{x}_t + Q\zeta_t^j : \zeta_t^j \sim N(0, I_4) \tag{4.3}$$

where Q is a diagonal matrix with the ith diagonal element denoted q_i. Each element in the signal vector is thus the sum of the true factor and an idiosyncratic noise component. The noise is uncorrelated across signals and time.

The vector $\mathbf{s}_t^j$ defined as

$$\mathbf{s}_t^j = \left[\begin{array}{ccc} \mathbf{z}_t^{j\prime} & r_t & \mathbf{y}_t' \end{array} \right]' \tag{4.4}$$

then contains all the signals that trader j observes in period t. Trader j's information set in period t also includes all previous signals

$$\Omega_t^j = \left\{ \mathbf{s}_t^j, \Omega_{t-1}^j \right\} \tag{4.5}$$

and traders thus condition their expectations on the entire history of observed signals.

4.4. **The law of motion of state.** When traders have heterogeneous information sets, it becomes optimal for them to form expectations about other traders' expectations. Natural representations of the state in this class of models tend to be infinite.[3] The model is solved using the method proposed in Nimark (2011) which delivers a law of motion for the (finite dimensional) state X_t of the form

$$X_t = MX_{t-1} + N\mathbf{e}_t. \tag{4.6}$$

The state vector X_t is given by the hierarchy of higher order expectations of the exogenous factors $\mathbf{x}_t$

$$X_t \equiv \left[\begin{array}{cccc} \mathbf{x}_t^{(0)\prime} & \mathbf{x}_t^{(1)\prime} & \cdots & \mathbf{x}_t^{(\overline{k})\prime} \end{array} \right]' \tag{4.7}$$

where the k order expectations is defined recursively as

$$\mathbf{x}_t^{(k)} \equiv \int E\left[\mathbf{x}_t^{(k-1)} \mid \Omega_t^j \right] dj$$

starting from $\mathbf{x}_t^{(0)} = \mathbf{x}_t$. The solution method in Nimark (2011) uses that the impact of higher order expectations on bond prices decreases "fast enough" in the order of expectation, and that the variance of higher order expectations is bounded by the variance of the true factors. Together, these facts imply that the equilibrium representation can be approximated with a state vector that contains only a finite number of higher order expectations of the factors. The integer $\overline{k}$ is the maximum order of expectation considered and can be chosen to achieve an arbitrarily close approximation in the limit as $\overline{k} \to \infty$. In the estimated model, $\overline{k} = 40$.

The vector $\mathbf{e}_t$ contains all the aggregate shocks that affect the extended state X_t and includes both the factor shocks $\mathbf{u}_t$ and the supply shocks $\mathbf{v}_t$. The supply shocks do not directly affect the factors $\mathbf{x}_t$ but they do affect traders' (higher order) expectations about $\mathbf{x}_t$ since traders use bond yields to extract information about $\mathbf{x}_t$.

[3]See Townsend (1983), Sargent (1991) and Makarov and Rytchkov (2012).

Common knowledge of the model among traders is used to pin down the law of motion for X_t, that is, to find M and N in (4.6). The logic is as follows: As usual in rational expectations models, first order expectations $\mathbf{x}_t^{(1)}$ are optimal, i.e. model consistent estimates of the actual factors $\mathbf{x}_t$. The knowledge that other traders have model consistent expectations allow traders to treat average first order expectations as a stochastic process with known properties when they form second order expectations. Common knowledge of the model thus implies that second order expectations $\mathbf{x}_t^{(2)}$ are optimal estimates of $\mathbf{x}_t^{(1)}$ given the law of motion for $\mathbf{x}_t^{(1)}$. Imposing this structure on all orders of expectations allows us to find the law of motion for the complete hierarchy of expectations as functions of the structural parameters of the model. The Appendix describes how to find the law of motion for the state in practice.

The state vector X_t is high dimensional, but this by itself does not increase our degrees of freedom in terms of fitting bond yields. In fact, because the endogenous state variables $\mathbf{x}_t^{(k)}$ are rational expectations of the lower order expectations in $\mathbf{x}_t^{(k-1)}$, the matrices M and N in the law of motion (4.6) are completely pinned down by the parameters of the process governing the true exogenous factors $\mathbf{x}_t$ and the precision of traders' information sets.

4.5. **Bond prices and the state.** For a given law of motion (4.6), bond prices can be derived using the average expectation operator $H : \mathbb{R}^{\overline{k}+1} \to \mathbb{R}^{\overline{k}+1}$ that annihilates the lowest order expectation of a hierarchy so that

$$\begin{bmatrix} \mathbf{x}_t^{(1)} \\ \mathbf{x}_t^{(2)} \\ \vdots \\ \mathbf{x}_t^{(\overline{k}+1)} \end{bmatrix} = H \begin{bmatrix} \mathbf{x}_t^{(0)} \\ \mathbf{x}_t^{(1)} \\ \vdots \\ \mathbf{x}_t^{(\overline{k})} \end{bmatrix} \tag{4.8}$$

and where $\mathbf{x}_t^{(k)} = 0 : k > \overline{k}$. The average (first order) expectation about the state in period t is thus given by HX_t. The average expectation in period t of what the state will be in period $t+1$ is thus given by MHX_t. Combing the operator H that increases the order of expectations by one step with the matrix M from the law of motion (4.6) that moves expectations one step forward in time, allows us to compute the k order expectation of the short rate in period $t+k$ as

$$r_{t:t+k}^{(k)} = \left[\begin{array}{cc} \delta_x & \mathbf{0} \end{array}\right] (MH)^{n-1} X_t. \tag{4.9}$$

Substituting (4.9) into the bond pricing equation (2.11) then gives

$$\begin{aligned} p_t^n \quad = \quad & \frac{1}{2}\sum_{i=2}^{n}\left(\sigma_i^2 - \Sigma_{rx}^i \boldsymbol{\mu}\right) - n\delta_0 \\ & -\sum_{s=0}^{n-1}\left[\begin{array}{cc} \delta_x & \mathbf{0} \end{array}\right](MH)^s X_t \\ & -v_t^n \end{aligned} \tag{4.10}$$

The matrix M governs the actual dynamics of r_t while bonds are priced *as if* X_t was observed by all traders and followed a process governed by MH. The matrices M and MH

are thus analogous to the "physical" and "risk neutral" dynamics in a standard no-arbitrage framework, though the interpretation is different.

4.6. **The estimated state space system.** The state equation (4.6) and the bond price equation (4.10) can be combined into a state space system of the form

$$X_t = MX_{t-1} + N\mathbf{e}_t \tag{4.11}$$

$$\overline{\mathbf{y}}_t = \overline{A} + \overline{B}X_t + \overline{R}\mathbf{v}_t. \tag{4.12}$$

Combining the fact that $y_t^n = -n^{-1}p_t^n$ with (4.10) implies that the rows of $\overline{A}$ and $\overline{B}$ that correspond to the n period bond yield in the measurement equation are given by

$$A_n = -n^{-1}\frac{1}{2}\sum_{i=2}^{n}\left(\sigma_i^2 - \Sigma_p^i\boldsymbol{\mu}\right) + \delta_0 \tag{4.13}$$

$$B_n = n^{-1}\sum_{s=0}^{n-1}\left[\begin{array}{cc}\delta_x & \mathbf{0}\end{array}\right](MH)^s X_t. \tag{4.14}$$

The vector of parameters to be estimated is denoted $\theta \equiv \{F, C, Q, \delta_0, \boldsymbol{\mu}, \lambda, \sigma_v\}$ and consists of a total of 22 parameters. Evaluating the log likelihood function for the state space system (4.11) - (4.12) allows us to form a posterior estimate for θ. The yields used for estimation are the 1-, 2-, 3-, 4- and 5-year interest rates on US Treasuries taken from the CRSP data base. The sample period runs from July 1952 to January 2013 and contains 727 monthly observations.

We use uniform priors on all model parameters. To take into account the evidence from the *Survey of Professional Forecasters* cited in the introduction, an informative prior is used on the model implied forecast dispersion. The prior distribution of the standard deviation of the cross-sectional forecast dispersion is centered around 20 basis points with a standard deviation of 5 basis points. This ensures that a low posterior probability is associated with parameterizations that imply either counter-factually small or implausibly large degrees of forecast dispersion among the traders in the model.

The posterior parameter distributions was generated from 200 000 draws from an Adaptive Metropolis algorithm (see Haario, Saksman and Tamminen 2001), initialized from a parameter vector found by maximizing the posterior using the simulated annealing maximizer of Goffe (1996). The results reported in the next section are based on the last 100 000 draws.

5. Empirical results

Table 1 reports the posterior estimates of the model parameters. The mode $\widehat{\theta}$ is the parameter vector from the Markov chain that achieves the highest posterior likelihood. All parameters appear to be well-identified.

The model fits unconditional yields well. At the posterior mode, the unconditional risk-free short rate is 5.90 per cent and the unconditional 5-year yield is 6.25 compared to the respective sample means of 5.24 and 5.73 per cent. The models thus slightly over-predicts unconditional yields but is able to generate an upward sloping yield curve. The upward slope is driven entirely by the covariance structure of conditional returns. In fact, the posterior mode of λ is 0.94. A value of λ smaller than 1 implies that the average supply of bonds is decreasing in maturity which by itself would generate a downward sloping yield curve.

The supply shocks have a standard deviation of 83 basis points. This is larger than the pricing errors usually found using yields-only affine no-arbitrage models. However, the supply shocks are not formally equivalent to the pricing errors of standard factor models such as those in the model of Joslin, Singleton and Zhu (2011). First, the supply shocks change traders' required compensation for risk and captures variation in common risk premia. The supply shocks are thus priced factors and not pricing errors.

Second, traders use the observed bond yields to extract information about the state X_t. Because the supply shocks affect bond yields, supply shocks affect traders (higher order) expectations about the persistent factors $\mathbf{x}_t$. So while supply shocks are independent across time and maturities, a supply shock to a single maturity bond has persistent effects on the entire cross-section of bond yields. The supply shocks thus have quite different observable implications compared to classical white noise pricing errors. That a single supply shock have persistent effects on yields across all maturities also means that the model is not subject to Hamilton and Wu's (2011) critique that classical measurement errors in affine term structure models can be statistically rejected.

At the posterior mode, the model implied cross-sectional dispersion of forecasts across the traders is approximately 10 basis points. While by itself, this level of dispersion is neither too large nor too small to appear a priori unreasonable, it is somewhat lower than the prior and also lower than what is found in survey data. That the posterior dispersion is lower than the prior and the dispersion measured in surveys suggest that conditional on the model, there is a trade-off between fitting bond yields and the cross-sectional dispersion. One possible interpretation of this result is that traders in reality are better and more uniformly informed than survey respondents and this may be inferable from bond yield dynamics.

Table 1
Posterior Parameter Estimates 1964:1-2007:12

θ	Mode $\widehat{\theta}$	Prior dist.	Posterior 2.5%-97.5%
		Short rate process	
δ_0	0.059	$U(0,\infty)$	0.058 - 0.061
ρ_1	0.99	$U(0,0.999)$	0.98 - 0.99
ρ_2	0.96	$U(0,0.999)$	0.95 - 0.98
ρ_3	0.70	$U(0,0.999)$	0.67-0.72
ρ_4	0.022	$U(0,0.999)$	0.020-0.027
c_1	0.012	$U(0,\infty)$	0.011- 0.013
c_2	0.0037	$U(0,\infty)$	0.0032 - 0.0038
c_3	0.012	$U(0,\infty)$	0.010 - 0.013
c_4	0.010	$U(-\infty,\infty)$	0.0094-0.013
c_{21}	-0.0015	$U(-\infty,\infty)$	$(-0.0017)-(-0.0013)$
c_{31}	-0.0019	$U(-\infty,\infty)$	$(-0.0020)-(-0.0090)$
c_{32}	-0.0045	$U(-\infty,\infty)$	$(-0.0049)-(-0.0043)$
c_{41}	0.00026	$U(-\infty,\infty)$	0.00024-0.00034
c_{42}	-0.020	$U(-\infty,\infty)$	$(-0.021)-(-0.0019)$
c_{43}	0.0067	$U(-\infty,\infty)$	0.0066 - 0.0068
		Noise in private signals	
q_1	0.011	$U(0,\infty)$	0.010 -0.012
q_2	0.015	$U(0,\infty)$	0.013-0.017
q_3	0.0027	$U(0,\infty)$	0.0025-0.0028
q_4	0.0014	$U(0,\infty)$	0.0013-0.0016
		Bond supply	
μ	0.50	$U(0,\infty)$	0.49 - 0.51
λ	0.94	$U(0,\infty)$	0.94-0.96
σ_v	0.0086	$U(0,\infty)$	0.0082 - 0.0088

5.1. **Yields and speculation.** We can use the estimated model to inspect the joint responses of yields, the speculative component and the speculative portfolio to innovations to the exogenous factors. The top row of Figure 1 illustrates the response of the 1- , 3- and 5-year bond yields. The middle row illustrates the response of the speculative component defined as (3.10) in the 1- , 3- and 5-year bond yields. The bottom row illustrates the speculative portfolio in 1- , 3- and 5-year bonds of the average trader defined as (3.8). Factors are ordered according to persistence with the impulse response functions to the most persistent factor in first column. The signs of the innovations are normalized so that the initial impact on yields is positive.

A few patterns stand out. First, short bond yields responds more to innovations than long bond yields. The magnitude of the response of the speculative term is uniformly larger in long maturity bonds than in shorter maturities. Conditional on the innovation, the speculative term responds in the same direction across all maturities. However, while an innovation to the first factor imply a positive response of the speculative component, innovations to the remaining factors imply negative responses. After an innovation to the most persistent factor, the average trader thus believes that other traders overestimate future short interest rates, while after an innovation to a less persistent factors, the average trader believes that other traders underestimate future short interest rates.

Inspecting the bottom row of Figure 1 shows that the speculative portfolio responses are stronger for 3-year bonds than for 1- and 5-year bonds. So while the speculative term is largest for long maturity bonds, the speculative position of the average traders is larger in medium maturity bonds. This may seem counterintuitive. However, the speculative portfolio (3.8) depends only on the difference between first and second order expectations of the one-period return while the speculative component depends on the accumulated difference between first and higher order expectations of future short rates over the entire life of the bond.

The speculative portfolio is a counter-factual thought experiment, and does not represent an actual change in bond demand in equilibrium. By definition, the average trader holds the average portfolio, which is unaffected by innovations to $\mathbf{x}_t$ so there are no actual changes in the average portfolio in response to the innovations plotted in Figure 1. However, the average trader is generally unaware of being the average trader. From his subjective perspective, the number of bonds that he holds because he thinks that average return expectations are incorrect do in fact respond to factor innovations.

Quantitatively, the speculative positions are large. This is due partly to the low degree of risk-aversion implied by the logarithmic preferences in expected wealth. Another, and more important, explanation to the large speculative positions is the strong correlation between returns on bonds of different maturities. A large long position in a particular maturity bond intended to exploit an expected positive excess return can then be hedged effectively by taking a large off-setting short positions.

5.2. **Historical decomposition of bond yields.** Proposition 2 above established that the speculative term in the price of a bond can be expressed as a higher order prediction error that is orthogonal to public information. Nevertheless, as econometricians, we can quantify this term using public bond price data since the period t higher order prediction error is only orthogonal to information known to all traders up to period t. Since we can ex post use the full sample and exploit information for $t+s : s > 0$ to back out information about the higher order prediction error in period t, we can form an estimate of the speculative component.

The procedure is as follows. For a given parameter vector θ, the Kalman simulation smoother can be used to draw from the smoothed state distribution $p\left(X^T \mid \mathbf{y}^T, \theta\right)$ (e.g. Durbin and Koopman 2002). To construct the posterior distribution of the state X^T, draw repeatedly from the posterior parameter distribution $p\left(\theta \mid \mathbf{y}^T\right)$ and for each draw of θ generate a draw from the conditional state distribution $p\left(X^T \mid \mathbf{y}^T, \theta\right)$. Since the speculative term (3.10) can be expressed as a linear function of the state X_t, the simulated distribution of the state can be used to compute the implied posterior distribution of the speculative component in bond yields.

Agents' average first order expectations of future risk-free rates are also linear functions of the state. Once we have a posterior distribution of the state we can thus construct a posterior distribution of the decomposition (3.12) and quantify how much the terms due to

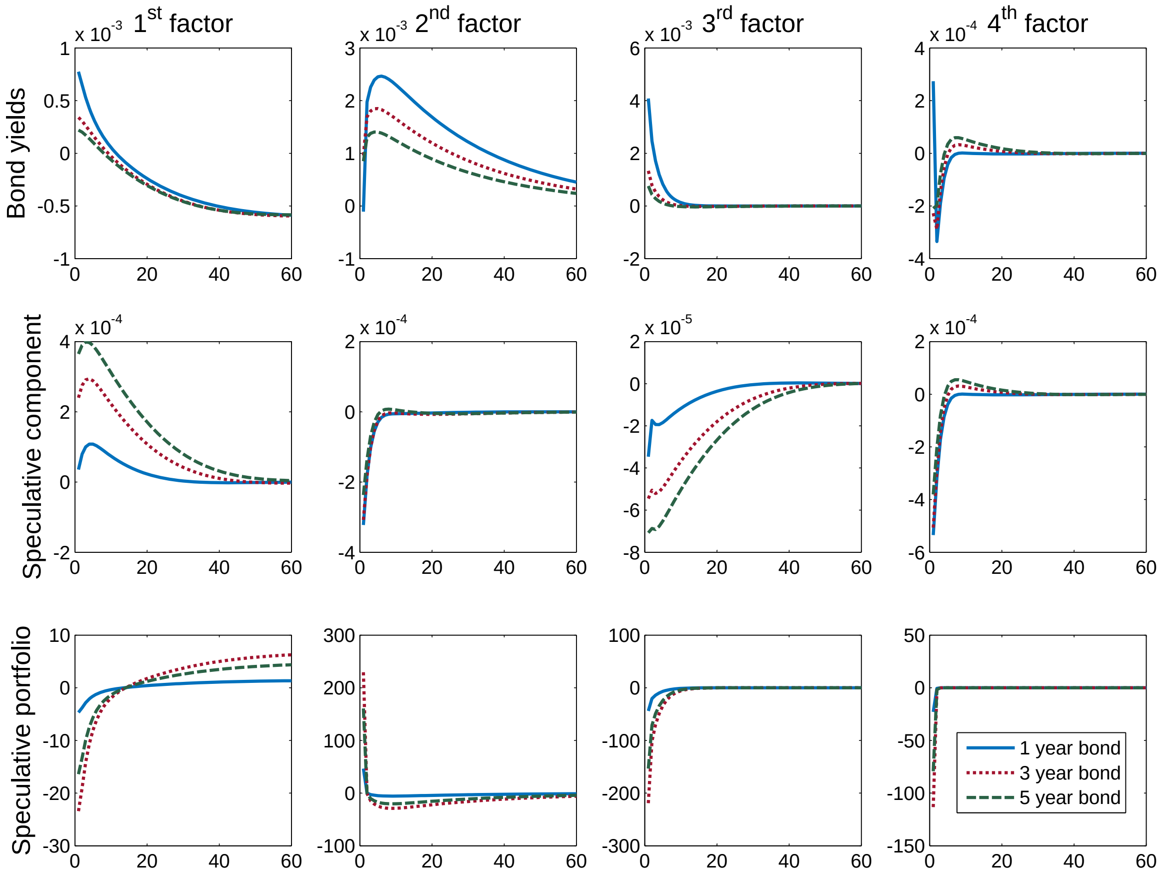

FIGURE 1. Impulse response function of 1,3 and 5 year yields (top row), the speculative component (middle row) and the speculative portfolio (bottom row) to innovations to the exogenous factors (by columns).

average first order expectations about future risk free rates, common risk premia and the speculation each contributed to bond yields over the sample period. Figure 2 and Figure 3 illustrate this decomposition for 1- and 5-year bond yields.

As in standard models, most of the variation in bond yields is explained by variation in expected future risk-free rates. Risk premia are positive on average and most volatile around the early 1980s for both the 1- and 5-year yield.

The speculative terms are positively correlated across maturities and more volatile in the 5-year bond than in the 1-year bond. The largest variations in the speculative term occur around 1980 when the speculative component for the 5-year bond contributes 120 negative basis points to the 5-year yield. This period coincides with the so-called Volcker disinflation when the then Federal Reserve chairman Paul Volcker raised interest rates sharply to bring inflation under control, causing a recession (see for instance the account in Goodfriend and King 2005). A negative speculative component indicates that traders thought other traders underestimated future short rates. Stated differently, the episode around 1980 during which

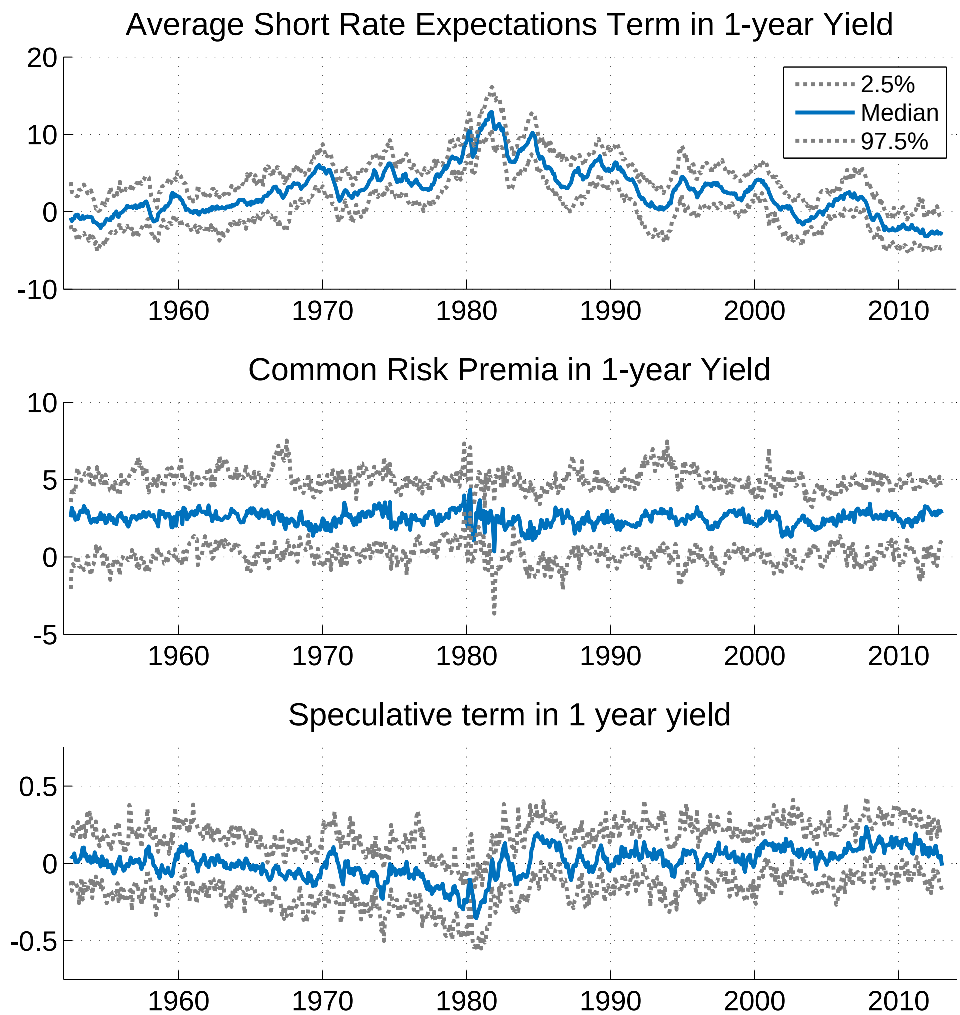

FIGURE 2. Historical decomposition of 5 year yield, median (solid) and 95% probability interval (dotted).

the speculative component is large and negative was a period when individual traders perhaps believed that other traders attached too much credibility to chairman Volcker's disinflation policy and were individually more sceptical about the probability of his eventual success.

In absolute terms, the speculative term was largest in the early 1980s. However, as a fraction of the total yield, speculation appears to have been more important in the last decade. In 2011, the mode of the estimate of the speculative term reached 40 basis points at a time when 5-year yields were around 3 per cent.

5.3. **Speculation and the expectations hypothesis.** One way to think of the well-known failure of the expectations hypothesis is in terms of a yield decomposition: If expectations of future short-rates are not sufficient to explain the variation in bond yields, the expectations

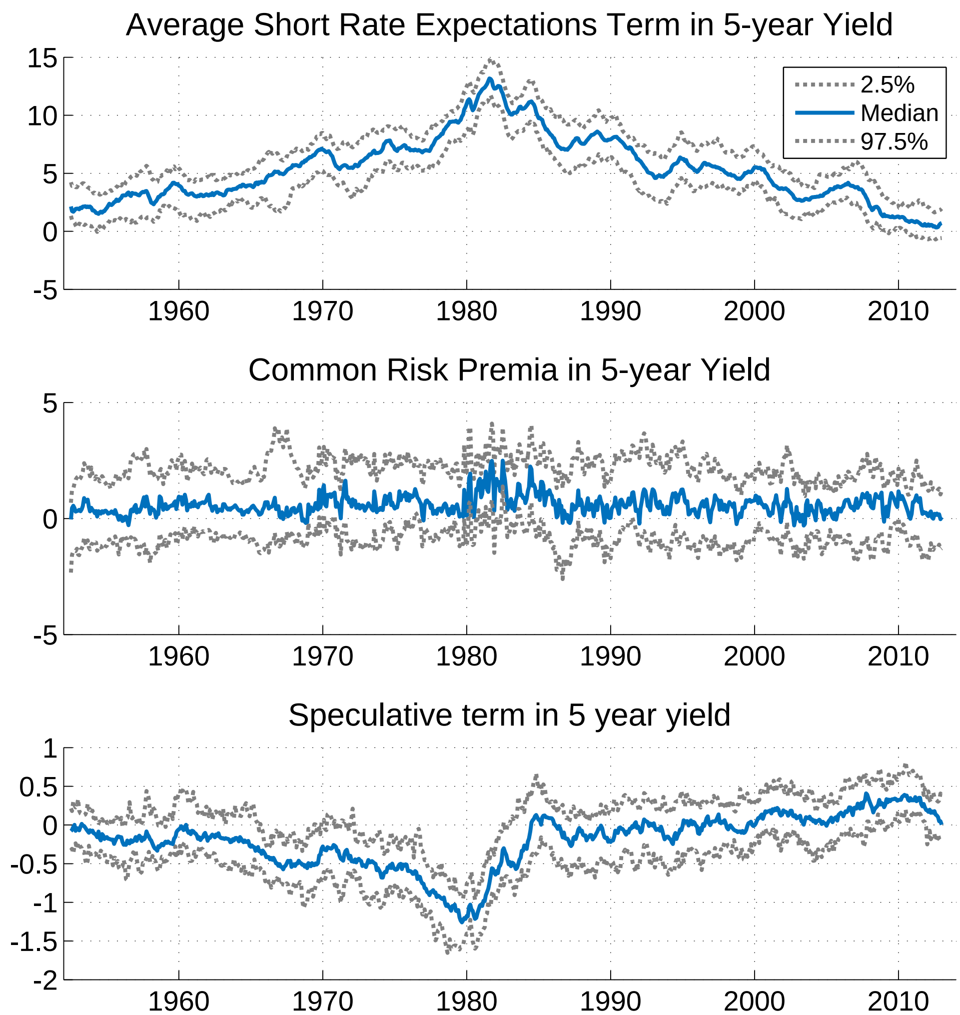

FIGURE 3. Historical decomposition of 5 year yield, median (solid) and 95% probability interval (dotted).

hypothesis fails. In this sense, the speculative component help to explain the failure of the expectations hypothesis since it provides a second wedge, in addition to classical risk-premia, between bond yields and expectations of future short rates.

A second way to think about the failure of the expectations hypothesis is in terms of excess returns being predictable, which is another way of stating that expectations of short-rates are not enough to explain bond yields. In this sense, the speculative component does not help explaining the well-documented empirical regularity that future excess returns are predictable based on the current yield curve (as well as many other variables). This is so because the speculative component must be orthogonal to publicly available information in real time.

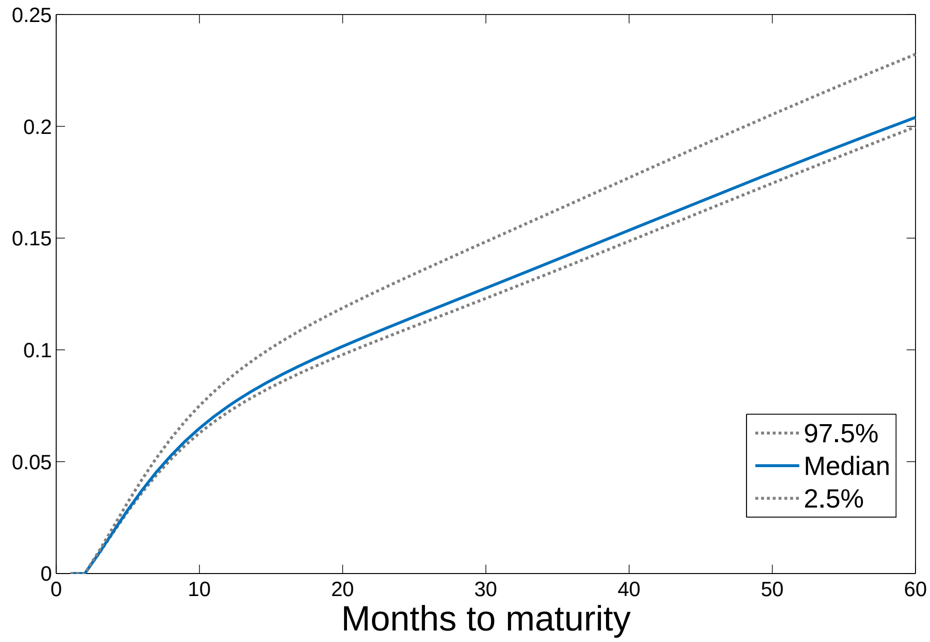

Figure 4. Relative standard deviation of speculative term and yields across maturities. Median (solid) and 95% probability interval (dotted).

Singleton (2006) points out that violations of the expectation hypothesis in US data are most pronounced when the period 1979-1983 is included in sample. Risk-premia based explanations of this episode emphasize that the early 1980s was a period when traders demanded either more compensation to hold a given amount of risk because of the recession, or when the amount of risk was perceived to be higher than usual because of more volatile interest rates. This is also the case for our model, though it abstracts from persistent variation in common risk premia, which may be a source of misspecification. One concern might be that because of the restrictive way that common risk premia is introduced, the model here simply relabels some of what in reality is risk-premia as speculation. However, while the early 1980s are associated with large movements in both risk-premia and speculation, the two are not observationally equivalent. The fact that speculative dynamics must be orthogonal to public information in real time makes it econometrically distinct from other sources of time variation in bond yields. This is exploited in Barillas and Nimark (2014) who use an affine no-arbitrage model that allows for heterogeneous information to separately identify risk premia and speculation over the same sample period. That model nests a standard three factor affine model and attributes a similar quantitative importance to the speculative component to what is found here.[4]

[4]The affine no-arbitrage model in Barillas and Nimark (2014) imposes less economic structure than the equilibrium model presented here and is empirically more flexible. However, in that paper, the portfolio choice of traders is not modeled explicitly.

5.4. **Speculation and yield volatility.** One way to illustrate the quantitative importance of the speculative term in bond yields of different maturities is to compute its standard deviation relative to the standard deviation of yields. Figure 4 displays the ratio of the standard deviation of the speculative component and bond yields across the yield curve. At the median, the standard deviation of the speculative component relative to the standard deviation of the yield increases from just below 10 percent for 1-year bonds to more than 20 percent for the 5-year bond. The speculative term thus accounts for a substantial fraction of the variation in long bond yields. The fact that speculative dynamics appear to be relatively more important for longer maturities may also help explain the evidence in Gürkaynak, Sack and Swanson (2005), who argue that current macro models of the term structure have trouble explaining the "excess" variability of long bond yields. Embedding a heterogeneous information structure in a macro model may improve these models' ability to match the variance of long term yields.

6. Conclusions

A fundamental question in finance is what the economic forces are that account for the variation in asset prices and returns. In this paper we have argued that if agents have access to different information, expectations about future risk-free short rates and risk-premia may not be sufficient to explain bond yields. Instead, we proposed that a novel speculative term, driven by heterogeneously informed traders, can account for a substantial fraction of the variation in historical US bond yields along with the classic terms.

As a theoretical contribution, we developed a tractable term structure model and used it to demonstrate that trader specific excess returns, as well as the component in bond prices that is due to heterogenous information, must be orthogonal to public information in real time. This property makes the speculative component in bond prices identified here different from the excess returns that can be predicted conditional on past bond yields, e.g. Fama and Bliss (1987) and Campbell and Shiller (1991). The speculative component is thus statistically distinct from the two classical components of the yield curve: risk-premia and terms reflecting expectations about future short rates.

Because traders in our model form rational expectations, the speculative component estimated here is distinct not only from the classical components of the term structure, but also from the speculative component in the difference-in-beliefs model of Xiong and Yan (2010). They propose that speculation among boundedly rational traders can provide an alternative explanations for the widely documented time-variation in predictable excess returns. In their model, the speculative term would to an outside econometrician looks like classic risk-premia. We do not take a stand here on the relative plausibility of rational versus boundedly rational traders, but from an econometric identification perspective, that speculation in our model has statistical properties that distinguishes it from classical risk premia likely also helps us to identify it sharply in the data.

While the empirical results of the paper suggest that speculative dynamics can be quantitatively important, the model is also quite restrictive. However, the fact that the speculative term must be orthogonal to public information available to traders in real time makes it difficult, or perhaps impossible, to use less model-dependent regression based strategies to quantify the importance of speculation among rational traders. One methodological contribution of the paper is thus to demonstrate how a structural approach can be used to estimate a historical time series of the effect of speculation on bond yields using publicly available bond yield data.

In a closely related paper, Barillas and Nimark (2014) estimate an affine no-arbitrage model with heterogeneously informed agents. That model is empirically more flexible and nests a standard affine no-arbitrage model as a special case while also allowing for the type of speculation analyzed here. Empirically, the speculative term extracted using that more flexible empirical model is qualitatively and quantitatively similar to what we found here.

The zero-coupon bonds traded in the model here have a known value at maturity. The uncertainty about future bond prices arise solely from the uncertainty about the discount rates that apply between the current period and the period when a bond matures. Arguably, these discount rates should matter also for the pricing of other assets that will be traded and pay dividends in the future. To the extent that there is additional uncertainty about dividend payments and returns on other classes of assets, speculative dynamics may be even more important in other markets than the bond market.

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Appendix A. Proof of Proposition 2

Proposition 2 *The speculative term $p_t^n - \overline{p}_t^n$ is orthogonal to public information in real time, i.e.*

$$E\left[p_t^n - \overline{p}_t^n \mid \Omega_t\right] = 0 \tag{A.1}$$

Proof. A typical element in the sum of higher order prediction errors

$$p_t^n - \overline{p}_t^n = \sum_{k=0}^{n-1}\left(\int E\left[r_{t+k} \mid \Omega_t^j\right] dj - r_{t:t+k}^{(k)}\right) \tag{A.2}$$

is can be written as

$$\int E\left[r_{t+k} \mid \Omega_t^j\right] dj - \int E\left[\int E\left[... \int E\left[r_{t+k} \mid \Omega_{t+k-1}^{j''}\right] dj'' ... \mid \Omega_{t+1}^{j'}\right] dj' \mid \Omega_t^j\right] dj \tag{A.3}$$

Taking expectations of both terms conditional on Ω_t^p gives

$$\begin{aligned} &E\left(\int E\left[r_{t+k} \mid \Omega_t^j\right] dj \mid \Omega_t\right) - \\ &E\left(\int E\left[\int E\left[... \int E\left[r_{t+k} \mid \Omega_{t+k-1}^{j''}\right] dj'' ... \mid \Omega_{t+1}^{j'}\right] dj' \mid \Omega_t^j\right] dj \mid \Omega_t\right) \end{aligned} \tag{A.4}$$

By Definition 1 the public information set is a subset of each trader's information set, i.e. that $\Omega_t \subseteq \Omega_t^j$ for each j. This fact, together with the law of iterated expectations implies that

$$\begin{aligned} &E\left(\int E\left[r_{t+k} \mid \Omega_t^j\right] dj \mid \Omega_t\right) - \\ &E\left(\int E\left[\int E\left[... \int E\left[r_{t+k} \mid \Omega_{t+k-1}^{j''}\right] dj'' ... \mid \Omega_{t+1}^{j'}\right] dj' \mid \Omega_t^j\right] dj \mid \Omega_t\right) \\ = \; &E\left(r_{t+k} \mid \Omega_t\right) - E\left(r_{t+k} \mid \Omega_t\right) \\ = \; &0 \end{aligned} \tag{A.5}$$

which completes the proof. □

Appendix B. Solving the model

Solving the model implies finding a law of motion for the higher order expectations of x_t of the form

$$X_{t+1} = MX_{t-1} + N\mathbf{e}_t \tag{B.1}$$

where

$$X_t \equiv \begin{bmatrix} x_t^{(0)} \\ x_t^{(1)} \\ \vdots \\ x_t^{(\overline{k})} \end{bmatrix}, \mathbf{e}_t = \begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_t \end{bmatrix}$$

That is, to solve the model, we need to find the matrices M and N as functions of the parameters governing the short rate process, the supply of long maturity bonds and the idiosyncratic noise shocks. The integer $\overline{k}$ is the maximum order of expectation considered and can be chosen to achieve an arbitrarily close approximation to the limit as $\overline{k} \to \infty$. Here, a brief overview of the method is given, but the reader is referred to Nimark (2011) for more details on the solution method.

First, common knowledge of the model can be used to pin down the law of motion for the vector X_t containing the hierarchy of higher order expectations of x_t. Rational, i.e. model consistent, expectations of x_t thus imply a law of motion for average expectations $\mathbf{x}_t^{(1)}$ which can then be treated as a new stochastic process. Knowledge that other traders are rational means that second order expectations $\mathbf{x}_t^{(2)}$ are determined by the average across traders of the rational expectations of the stochastic process $\mathbf{x}_t^{(1)}$. The average third order expectation $\mathbf{x}_t^{(3)}$ is then the average of the rational expectations of the process $\mathbf{x}_t^{(2)}$, and so on. Imposing this structure on all orders of expectations allows us to find the matrices M and N. How this is implemented in practice is described below.

Second, the method exploits that the importance of higher order expectations is decreasing in the order of expectations. This result has two components:

(i) The variances of higher order expectations of the factors $\mathbf{x}_t$ are bounded by the variance of the true process. More generally, the variance of $k+1$ order expectation cannot be larger than the variance of a k order expectation

$$cov\left(\mathbf{x}_t^{(k+1)}\right) \leq cov\left(x_t^{(k)}\right) \tag{B.2}$$

To see why, first define the average $k+1$ order expectation error $\zeta_t^{(k+1)}$

$$\mathbf{x}_t^{(k)} \equiv \mathbf{x}_t^{(k+1)} + \zeta_t^{(k+1)} \tag{B.3}$$

Since $\mathbf{x}_t^{(k+1)}$ is the average of an optimal estimate of $\mathbf{x}_t^{(k)}$ the error $\zeta_t^{(k+1)}$ must be orthogonal to $\mathbf{x}_t^{(k+1)}$ so that

$$cov\left(\mathbf{x}_t^{(k)}\right) = cov\left(\mathbf{x}_t^{(k+1)}\right) + cov\left(\zeta_t^{(k+1)}\right). \tag{B.4}$$

Now, since covariances are positive semi-definite we have that

$$cov\left(\zeta_t^{(k+1)}\right) \geq \mathbf{0} \tag{B.5}$$

and the inequality (B.2) follows immediately. (This is an abbreviated description of a more formal proof available in Nimark 2011.)

That the variances of higher order expectations of the factors are bounded is not sufficient for an accurate finite dimensional solution. We also need (ii) that the impact of the expectations of the factors on bond yields decreases "fast enough" in the order of expectation. The proof of this result is somewhat involved and interested readers are referred to the original reference.That is, to solve the model, we need to find the matrices M, N and B as functions of the parameters governing the short rate process, the stochastic supply shocks and the idiosyncratic noise shocks. The integer $\overline{k}$ is the maximum order of expectation considered and can be chosen to achieve an arbitrarily close approximation to the limit as $\overline{k} \to \infty$. Here, a brief overview of the method is given, but the reader is referred to Nimark (2010) for more details on the solution method.

B.1. **The law of motion of the state.** To find the law of motion for the hierarchy of expectations X_t we use the following strategy. For a given M, N and B in (4.11) - (4.12) we will derive the law of motion for trader j's expectations of X_t , denoted $X^j_{t|t} \equiv E\left[X_t \mid \Omega^j_t\right]$. First, write the vector of signals S^j_t as a function of the state, the aggregate shocks and the idiosyncratic shocks

$$\begin{aligned} S^j_t &= \left[\begin{array}{ccc} \mathbf{z}'^{\mathbf{j}}_{\mathbf{t}} & \widetilde{r}_t & \mathbf{y}'_{\mathbf{t}} \end{array}\right]' & \text{(B.6)} \\ &= \mu_S + DX_t + R\left[\begin{array}{c} \zeta^j_t \\ \mathbf{u}_t \\ \widehat{\mathbf{v}}_t \end{array}\right] & \text{(B.7)} \end{aligned}$$

where by (4.1), (4.3), (4.13) and (4.14) μ_S and D are given by

$$\mu_S = \left[\begin{array}{ccccc} \mathbf{0}_{1\times 4} & \delta_0 & A_2 & \cdots & A_{\overline{n}} \end{array}\right]' \tag{B.8}$$

and

$$D = \left[\begin{array}{cc} I_4 & \mathbf{0} \\ \mathbf{1}_{1\times 4} & \mathbf{0} \\ \multicolumn{2}{c}{B} \end{array}\right], B \equiv \left[\begin{array}{cccc} B'_1 & B'_2 & \cdots & B'_{\overline{n}} \end{array}\right]'. \tag{B.9}$$

The matrix R can be partitioned conformably to the idiosyncratic and aggregate shocks

$$R = \left[\begin{array}{cc} R_j & R_A \end{array}\right].$$

Trader j's updating equation of the state $X^j_{t|t}$ estimate will then follow

$$X^j_{t|t} = MX^j_{t|t-1} + K\left(S^j_t - \mu_S - DMX^j_{t|t-1}\right) \tag{B.10}$$

Rewriting the observable vector S^j_t as a function of the lagged state and taking averages across traders using that $\int \zeta^j_t dj = 0$ yields

$$\begin{aligned} X_{t|t} &= MX_{t|t-1} + K\left(DMX_{t-1} + (DN + R_A)\,\mathbf{e}_t + -DMX^j_{t|t-1}\right) & \text{(B.11)} \\ &= (M - KDM)\,X_{t|t-1} + KDMX_{t-1} + K\,(DN + R_A)\,\mathbf{e}_t & \text{(B.12)} \end{aligned}$$

Appending the average updating equation to the exogenous state gives us the conjectured form of the law of motion of $\mathbf{x}^{(0:\overline{k})}_t$

$$\begin{bmatrix} \mathbf{x}_t \\ X_{t|t} \end{bmatrix} = M \begin{bmatrix} \mathbf{x}_{t-1} \\ X_{t-1|t-1} \end{bmatrix} + N\mathbf{e}_t$$

where M and N are given by

$$\begin{aligned} M &= \begin{bmatrix} F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0} \\ \mathbf{0} & [M - KDM]_- \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ [KDM]_- \end{bmatrix} & \text{(B.13)} \\ N &= \begin{bmatrix} C & 0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ [K\,(DN + R_A)]_- \end{bmatrix} & \text{(B.14)} \end{aligned}$$

where $[\cdot]_{-}$ indicates that the a last row or column has been canceled to make a the matrix $[\cdot]$ conformable, i.e. implementing that $\mathbf{x}_t^{(k)} = 0 : k > \overline{k}$. The Kalman gain K in (B.10) is given by

$$K = (PD' + NR')\,(DPD' + RR')^{-1} \tag{B.15}$$

$$P = M\left(P - (PD' + NR')\,(DPD' + RR')^{-1}\,(PD' + NR')'\right)M' + NN' \tag{B.16}$$

The model is solved by finding a fixed point that satisfies (4.12), (B.13), (B.14), (B.15) and (B.16).