TOPICS IN MACROECONOMICS: MODELLING INFORMATION, LEARNING AND EXPECTATIONS

MIDTERM DECEMBER 3 2009

There are 4 questions. Question 1 should be answered by everybody. You should choose 2 more questions to answer from the remaining 3. Each answered question is worth a maximum of 10 points. Put your name on each sheet of paper that you hand in. Write clearly. Number the pages. Include a "front page" with your name, which questions you answered and how many pages you handed in. Sign the front page. Good luck.

Question 1: Basic methods, short answers welcome

a) Consider the two random vectors X and Y

$$X = a\theta + u \tag{0.1}$$
$$Y = b\theta + v \tag{0.2}$$

where a and b are constants, $E[\theta\theta'] = \sigma_\theta^2, E[uu'] = \sigma_u^2, E(u\theta') = 0, E[vv'] = \sigma_v^2, E[v\theta'] = 0$ and $E[vu'] = 0$. Find the minimum variance linear estimator of θ conditional on X and Y using the properties of orthogonal projections.

b) Let $\mathcal{P}(X|Z,Y)$ denote the orthogonal projection of X on the space spanned by Z and Y. Show that

$$\mathcal{P}(X|Z,Y) = \mathcal{P}(X|Z) + \mathcal{P}(X|Y) \tag{0.3}$$

if $\mathcal{P}(Y|Z) = 0$.

For part c), d) and e), consider the state space system

$$X_t = AX_{t-1} + C\mathbf{u}_t \tag{0.4}$$
$$Z_t = DX_t + \mathbf{v}_t \tag{0.5}$$

where X_t is an $n \times 1$ vector of random variables, $\mathbf{u}_t$ is an $m \times 1$ vector of i.i.d. shocks with unit variance, i.e. $E\left[\mathbf{u}_t\mathbf{u}'_{t+s}\right] = I$ if $s = 0$ and $\mathbf{0}$ otherwise. A and C are ($n \times n$ and $n \times m$ respectively) coefficient matrices. Z_t is an $(l \times 1)$ vector of observables and D is an $(l \times n)$ selector matrix that combines elements of the state X_t into observable variables and $\mathbf{v}_t$ is an $(l \times 1)$ vector of measurement errors with covariance Σ_{vv}.

c) Define $X_{t|t}$ as the linear minimum variance estimate of X_t conditional on the complete history of observables Z^t. What is $E\left[\left(X_t - X_{t|t}\right)X'_{t|t}\right]$?

d) Give an example of A, C, D and Σ_{vv} such that
$E\left[\left(X_t - X_{t|t}\right)\left(X_t - X_{t|t}\right)'\right] = E\left[\left(X_t - E\left[X_t \mid Z_t\right]\right)\left(X_t - E\left[X_t \mid Z_t\right]\right)'\right]$. Motivate.

e) If possible, order the positive semi-definite matrices $\mathbf{0}_{n\times n}$, Σ_{xx}, CC', $P_{t|t}$ and $P_{t|t-1}$, where $P_{t|t-1}$ and $P_{t|t-1}$ are respectively the prior and posterior covariance of the state estimate $X_{t|t}$ generated by the Kalman filter. Use that for two positive semi-definite matrices Q and R, the ordering relation $Q \geq R$ implies that $Q - R$ is a non-negative semi-definite matrix. Motivate.

Question 2: Endogenous Information Choice and Rational Inattention

a) Define entropy and mutual information.

b) Let the entropy of the n-dimensional vector X be $h(X)$ and let $Y = BX$ with $rank(B) = n$. What is the conditional entropy $h(X \mid Y)$? What is the mutual information $I(X;Y)$?

c) Solve for the optimal allocation of attention (i.e. choose posterior variances) in the following set up:

Expected loss

$$EU = \lambda^2 \sigma_1^2 + (1-\lambda)^2 \sigma_2^2 \tag{0.6}$$

subject to

$$\ln |\Sigma_{prior}| - \ln |\Sigma_{post}| \leq \kappa \tag{0.7}$$

where

$$\Sigma_{post} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \tag{0.8}$$

for

$$\Sigma_{prior} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \tag{0.9}$$

$$e^{-\kappa} = 1/10 \tag{0.10}$$

d) Define the "no forgetting constraint".

e) Find two marginal conditions for λ where the "no forgetting constraint" starts/stops binding.

Question 3: Private and public information

a) Consider the unobservable variable θ given by

$$\theta \sim N\left(0, \sigma_\theta^2\right) \tag{0.11}$$

Agents (indexed by j) observe a private noisy signal of θ given by

$$z(j) = \theta + \varepsilon(j) \tag{0.12}$$

$$\varepsilon(j) \sim N\left(0, \sigma_\varepsilon^2\right) \forall j \tag{0.13}$$

That is, all agents receive an equally precise signal of θ but agent j only observes his own signal $z(j)$.Define

$$\theta^{(k)} \equiv \int E\left[\theta^{(k-1)} \mid z(j)\right] dj \tag{0.14}$$

$$\theta^{(0)} \equiv \theta \tag{0.15}$$

Find an expression for $\theta^{(k)}$. What is the limit as $k \to \infty$?

b) Consider the set up above, but where the signal is instead given by

$$z(j) = \theta + \delta \quad \forall j : \delta \sim N\left(0, \sigma_\delta^2\right) \forall j \tag{0.16}$$

Find a new expression for $\theta^{(k)}$. Interpret.

c) Consider the model of Morris and Shin (AER 2002). Utility of agent $i \in (0,1)$ is given by

$$U_i = -(1-r)(a_i - \theta)^2 - r\left(L_i - \overline{L}\right) \tag{0.17}$$

where a_i is the action taken by agent i and

$$L_i = \int (a_j - a_i)^2 \, dj \tag{0.18}$$

and

$$\overline{L} = \int L_j dj \tag{0.19}$$

Agents observe two signals of θ. The public signal y

$$\begin{aligned} y &= \theta + \eta \\ \eta &\sim N(0, \sigma_\eta^2) \end{aligned} \tag{0.20}$$

and the private signal x_i

$$\begin{aligned} x_i &= \theta + \varepsilon_i \\ \varepsilon_i &\sim N(0, \sigma_\varepsilon^2) \forall i \end{aligned} \tag{0.21}$$

The first order condition for expected utility maximization is given by

$$a_i = (1 - r) E[\theta \mid x_i, y] + rE\left[\int a_j dj \mid x_i, y\right] \tag{0.22}$$

where $\int a_j dj$ ($\equiv \overline{a}$) is the average action across agents. Find κ in the optimal linear reaction function of agent i

$$a_i = \kappa x_i + (1 - \kappa) y \tag{0.23}$$

Solve for equilibrium average action $\overline{a}$ as a function of θ and η.

Expected welfare is decreasing in the precision of the public signal α in Morris and Shin (2002) if

$$\frac{\alpha}{\beta} < (2r - 1)(1 - r). \tag{0.24}$$

d) For what range of r can the inequality hold? Why? Interpret.

e) Discuss L.E.O. Svensson's argument about the value of transparency in relation to the model of Morris and Shin.

Question 4: Miscellaneous

You may use words, algebra or both in your answers.

The information revealed by markets. a) Explain intuitively Grossman and Stiglitz's result regarding the impossibility of informationally efficient markets when information is costly. Carefully define all terminology used.

b) Explain intuitively the bounds on the costs of information that guarantees that some agents will and some agents will not buy the signal, i.e. the bounds that guarantees an interior solution.

Learning and Bounded Rationality. c) Describe intuitively decreasing and constant gain learning.

d) Describe what the policy makers in the paper *The Conquest of U.S. Inflation: Learning and Robustness to Model Uncertainty* by Cogley and Sargent RED (2005) are learning about.

e) Describe how probabilities and expected losses of individual sub-models relates to the optimal policy in a Bayesian Robustness setting.