TOPICS IN MACROECONOMICS: MODELLING INFORMATION, LEARNING AND EXPECTATIONS

LECTURE NOTES 4

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Examples of Imperfect Information Models

These notes describes and discusses two imperfect information models: The famous Lucas Island model (Lucas 1972, 1973,1975) and a simplified version of the less famous, but never the less illustrative RBC model with noisy indicators of Bomfim (2001).

1. Lucas Island Model

The Lucas Island model appeared in a series of papers in the early 1970s (see Lucas 1972, 1973, 1975). These papers became influential for several reasons; they demonstrated the "natural" rate hypothesis and the (long run) neutrality of money in a rigorous setting, they popularized rational expectations and they changed the style of macro economics by building aggregate models on a foundation of optimizing behaviour by individuals. What is not always remembered is that business cycles in the island model stems from individuals misperceptions about relative prices, an idea that was almost dead for 20 years from the early 80s to the early 00's. Recently, there has been a revival of the idea that the dynamics of business cycle fluctuations can be influenced by information imperfections, e.g. Mankiw and Reis (2002), Woodford (2002), Mackowiack and Wiederholt (2008) and Lorenzoni (2007).

The models in Lucas's papers differ somewhat in set ups and complexity, and here we will focus on perhaps the most accessible version, that of the 1973 AER paper. The basic idea is the following. Supply (and production) is determined by expected relative prices; when

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producers expect a high relative price of the good produced on their island, they produce more of it. However, supply decisions are made based on partial information. The nominal price of the good produced on each island is observed only on that island, and the aggregate price level is observed only with a lag. The problem facing the producers on each island is thus to figure out how much of the change in their own good's price reflects a general price change and how much reflects a change in relative prices?

Lucas used a prediction from the model to test his hypothesis: in countries where nominal demand varies a lot, producers are more likely to attribute changes in prices to aggregate changes rather than relative price changes. This observation has implications for regression coefficients that are also born out in his empirical exercise.

Here is the model.

1.1. **Supply.** There is a continuum of islands indexed by z. Supply on island z is given by

$$y_t(z) = y_{nt} + y_{ct}(z) \tag{1.1}$$

where y_{nt} is the natural, or trend, component of supply and $y_{ct}(z)$ is the cyclical component. The trend component follows the deterministic law of motion

$$y_{nt} = \alpha + \beta t \tag{1.2}$$

We will mostly be concerned with the cyclical component of supply. Supply on island z is given by

$$y_{ct}(z) = \gamma \left[P_t(z) - E\left(P_t \mid I_t(z) \right) \right] + \lambda y_{c,t-1}(z) \tag{1.3}$$

Lucas sets up the model to make the signal extraction as simple as possible: he assumes that lagged price levels and aggregate output deviations from trend are observable. This result in a common prior for the mean and variance of the aggregate price level

$$P_t \sim N\left(\overline{P}_t, \sigma^2 \right) \tag{1.4}$$

The price in island z is (exogenously) given by

$$P_t(z) = P_t + z \tag{1.5}$$

where $z \sim N(0, \tau^2)$. The supply of good z is then given by

$$y_{ct}(z) = \gamma\theta\left[P_t(z) - \overline{P}_t\right] + \lambda y_{c,t-1}(z) \tag{1.6}$$

where

$$\theta = \frac{\tau^2}{\tau^2 + \sigma^2} \tag{1.7}$$

and

$$\sigma^2 = E\left[P_t - \overline{P}_t\right]^2 \tag{1.8}$$

The intuition behind the supply curve (1.6) is that production will be higher on islands with a high expected relative price compared to the aggregate price level.

1.2. **Demand.** (The log of) nominal demand is postulated as

$$y_t + P_t \;=\; x_t \tag{1.9}$$

$$\Delta x_t \;\equiv\; (x_t - x_{t-1}) \sim N\left(\delta, \sigma_x^2\right) \tag{1.10}$$

1.3. **Information sets.** Lucas assumes that all agents observe the complete history of the aggregate price level and past shocks to demand up to the last period. The only current information available to island z is the price of the good produced on their own island, $P(z)$. The information set at time t on island z is thus given by

$$I_t(z) = \{p_{t-s+1}(z), P_{t-s}, x_{t-s}, y_{t-s} : s = 1, 2, ...\} \tag{1.11}$$

1.4. **Solving the model.** All the action in the model comes from agents misperceptions about the relative price of the good produced on their own island. Without loosing much of interest, we can therefore assume that all constants and deterministic variables are zero, i.e.

that $\delta = y_{nt} = 0$, and that λ, the parameter governing the importance of lagged supply, is zero since these only affect the predictable components of output and prices. What remains are the following relationships: An aggregate supply schedule

$$y_t = \theta\gamma \left[P_t - \overline{P}_t\right] \tag{1.12}$$

which can be found by averaging across island specific demand schedules. Aggregate (real) demand can be found by rearranging (1.9) to get

$$y_t = x_t - P_t \tag{1.13}$$

To solve the model, we start by conjecturing a solution to P_t and $\overline{P}_t$ of the form

$$P_t = \pi_1 x_t + \pi_2 x_{t-1} \tag{1.14}$$

and

$$\overline{P}_t = \pi_1 E_{t-1}\left[x_t\right] + \pi_2 x_{t-1} \tag{1.15}$$

$$= \pi_1 x_{t-1} + \pi_2 x_{t-1} \tag{1.16}$$

Then combine (1.12) and (1.13) to get

$$x_t - P_t = \theta\gamma \left[P_t - \overline{P}_t\right] \tag{1.17}$$

and then use (1.14) and (1.15) to substitute out P_t and $\overline{P}_t$

$$x_t - (\pi_1 x_t + \pi_2 x_{t-1}) = \theta\gamma\left(\pi_1 x_t + \pi_2 x_{t-1}\right) - \theta\gamma(\pi_1 + \pi_2)x_{t-1} \tag{1.18}$$

Equating coefficients gives

$$1 - \pi_1 = \theta\gamma\pi_1 \tag{1.19}$$

and

$$-\pi_2 = \theta\gamma\pi_2 - \theta\gamma(\pi_1 + \pi_2). \tag{1.20}$$

Solve for π_1 and π_2

$$\pi_1 = \frac{1}{1+\theta\gamma} \tag{1.21}$$

$$\pi_2 = \frac{\theta\gamma}{1+\theta\gamma} \tag{1.22}$$

We can then plug these expressions back into the conjectured solutions

$$P_t = \frac{1}{1+\theta\gamma}x_t + \frac{\theta\gamma}{1+\theta\gamma}x_{t-1} \tag{1.23}$$

$$\overline{P}_t = \left(\frac{1}{1+\theta\gamma} + \frac{\theta\gamma}{1+\theta\gamma}\right)x_{t-1} \tag{1.24}$$

Substituting in the result in the supply curve (1.12)

$$y_t = \theta\gamma\left[P_t - \overline{P}_t\right] \tag{1.25}$$

$$= \frac{\theta\gamma}{1+\theta\gamma}\Delta x_t \tag{1.26}$$

We can then also solve for θ as

$$\theta = \frac{\tau^2}{\tau^2 + \sigma^2} \tag{1.27}$$

where $\sigma^2 \equiv E\left[P_t - \overline{P}_t\right]^2$ is given by

$$\sigma^2 = \frac{1}{(1+\theta\gamma)^2}\sigma_x^2 \tag{1.28}$$

This is not a closed form solution for θ, but it can still be used to illustrate several properties of the model.

1.5. **Testable properties of the model.** Changes in the log price level, i.e. inflation, is given by

$$\Delta P_t = \frac{1}{1+\theta\gamma}\Delta x_t + \frac{\theta\gamma}{1+\theta\gamma}\Delta x_{t-1} \tag{1.29}$$

$$= (1-\pi)\,\Delta x_t + \pi \Delta x_{t-1} \tag{1.30}$$

where

$$\pi = \frac{\tau^2\gamma}{(1-\pi)^2\,\sigma_x^2 + \tau^2\,(1+\gamma)} \tag{1.31}$$

from the definition (1.27) of θ. The significance of this expression is that it predicts that π should be small for countries where the variance $\sigma_x^2$of changes in nominal demand Δx_t is high, since π tends monotonically towards zero as σ_x^2 gets larger.

Lucas runs the regressions

$$\Delta P_t = \beta_1 \Delta x_t + \beta_2 \Delta x_{t-1} + \varepsilon_t^p \tag{1.32}$$

$$y_{ct} = \beta_3 \Delta x_t + \varepsilon_t^y \tag{1.33}$$

and confirms the model's predictions that β_2 and β_3 should be a small number for countries with highly variable nominal output.

1.6. **The variance of real output.** Real output volatility is increasing in the variance of island specific price variance, as

$$E\left[y_{ct}y_{ct}\right] = \frac{\theta\gamma}{1+\theta\gamma}\sigma_x^2$$

and θ is increasing in τ^2. Also, since $\tau^2 = 0 \implies \theta = 0$, it also implies that $\tau^2 = 0 \implies E\left[y_{ct}y_{ct}\right] = 0$.

2. A simple RBC model with noisy indicators

In a paper from 2001, Antulio Bomfim proposes to quantify the effect of noisy statistical releases, using a version of a Real Business Cycle (RBC) model. Below a simplified version of Bomfim's model, drawing on Campbell (1994), is presented. It is a useful exercise in how to solve a model where agents are assumed to use the Kalman filter to extract an estimate of an unobservable state. The model differs from Lucas' in both substantive and methodological content. Particularly, in Bomfim's model, output (and consumption) variability increase with information precision, and the filtering problem of the agents is dynamic as the state is not revealed with a one period lag.

2.1. **The model.** Output is given by the production function

$$Y_t = (A_t N_t)^{\alpha} K_t^{1-\alpha} \tag{2.1}$$

where $A_{t,}N_t$ and K_t are productivity, labour input and capital stock, respectively. The law of motion for capital stock is given by

$$K_{t+1} = (1-\delta) K_t + Y_t - C_t \tag{2.2}$$

where δ is the depreciation rate and C_t is consumption, chosen to maximize

$$U = E\left[\sum_{s=0}^{\infty} \beta^s \frac{C_{t+s}^{1-\gamma}}{1-\gamma} \mid Z^t\right] \tag{2.3}$$

We will assume that labour supply is fixed so that $N_t = 1 \forall t$. C_t is then the only choice variable and thus the only variable determined by expectations. $Z^t \equiv \{Z_t, Z_{t-1}, ...\}$ is the history of the vector of observables (specified below) up to time t.

2.2. **The log-linearized model.** We will use the convention that lower case letters denote log deviations from steady state values of corresponding upper case letters. (Log deviations of) output then is

$$y_t = \alpha a_t + (1-\alpha) k_t \tag{2.4}$$

where (log of) productivity a_t is the sum of a persistent and transitory component

$$a_t = a_{1,t} + a_{2,t} \tag{2.5}$$

where

$$a_{1,t} = \rho a_{1,t-1} + u_{1,t} \tag{2.6}$$

$$u_{1,t} \sim N\left(0, \sigma_1^2\right) \tag{2.7}$$

$$a_{2,t} \sim N\left(0, \sigma_2^2\right) \tag{2.8}$$

The capital stock evolves according to

$$k_{t+1} = \lambda_1 k_t + \lambda_2 a_t + \lambda_3 c_t \tag{2.9}$$

$$\lambda_3 = 1 - \lambda_1 - \lambda_2 \tag{2.10}$$

2.3. **The full information solution.** Start by finding the solution to the full information model. It will be of the form

$$\begin{bmatrix} k_t \\ a_{1,t} \\ a_{2,t} \end{bmatrix} = M \begin{bmatrix} k_{t-1} \\ a_{1,t-1} \\ a_{2,t-1} \end{bmatrix} + N \begin{bmatrix} u_{1,t} \\ a_{2,1} \end{bmatrix} \tag{2.11}$$

$$c_t = \mathbf{g}' \begin{bmatrix} k_t \\ a_{1,t} \\ a_{2,t} \end{bmatrix} \tag{2.12}$$

$$\mathbf{g}' = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} \tag{2.13}$$

2.4. **Solving the model under imperfect information.** Bomfim assumes that agents cannot separate the two components of productivity by direct observation. However, the decomposition of productivity into the persistent and transitory component matters since it will influence expectations about future productivity, and therefore consumption.Bomfim assumes that agents observe a noisy measure π_t of the sum of the two components as well as aggregate output y_t. The observation vector Z_t is thus given by

$$Z_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} \tag{2.14}$$

$$= DX_t \tag{2.15}$$

where

$$X_t = \begin{bmatrix} k_t \\ a_{1,t} \\ a_{2,t} \\ e_t \end{bmatrix} \tag{2.16}$$

$$D = \begin{bmatrix} (1-\alpha) & \alpha & \alpha & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \tag{2.17}$$

We have thus included the noise term e_t in the state X_t. The reason is that the noise term will affect the capital stock indirectly , through its effect on consumption. Putting the measurement errors in the state is sometimes easier when noise shocks are correlated state and comes at small computational cost when the state is of low dimension.

In models where agents have an imperfect estimate of the state, we need to expand the state to include not only the "true" state of the model but also agent's estimate of the state.

We want find a solution of the following form

$$\begin{bmatrix} X_t \\ X_{t|t} \end{bmatrix} = W \begin{bmatrix} X_{t-1} \\ X_{t-1|t-1} \end{bmatrix} + V\varepsilon_t \tag{2.18}$$

where

$$\varepsilon_t = \begin{bmatrix} u_{1,t} \\ a_{2,1} \\ e_t \end{bmatrix}, \qquad E\left[\varepsilon_t \varepsilon_t'\right] = \Sigma_{\varepsilon\varepsilon}$$

The solution to the upper half of W

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{2.19}$$

is then given by

$$W_{11} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.20}$$

$$W_{12} = \lambda_3 \begin{bmatrix} g_1 & g_2 & g_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.21}$$

since certainty equivalence dictates that $c_t = \mathbf{g}' X_{t|t}$. Plugging this into the law of motion for capital (2.9) yields W_{11} and W_{12}.

2.4.1. *The law of motion of the state estimate.* To completely solve the system, we also need to find the law of motion for the estimate of the state $X_{t|t}$. It is given by the Kalman filter updating equation

$$X_{t|t} = (W_{11} + W_{12})\, X_{t-1|t-1} + K\left[Z_t - D\,(W_{11} + W_{12})\, X_{t-1|t-1}\right] \tag{2.22}$$

First step is to use that the vector of observables Z_t is given by

$$Z_t = DW_{11}X_{t-1} + DW_{12}X_{t-t|t-1} + DV_{11}\varepsilon_t$$

where

$$V_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting this into the updating equation (2.22) and simplifying yields

$$X_{t|t} = (W_{11} + W_{12})\, X_{t-1|t-1} + K\left[DW_{11}X_{t-1} + DV_{11}\mathbf{e}_t - DW_{11}X_{t-1|t-1}\right] \tag{2.23}$$

We can now construct the matrices W and V from (2.18) as

$$\begin{bmatrix} X_t \\ X_{t|t} \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ KDW_{11} & (W_{11}+W_{12}) - KDW_{11} \end{bmatrix} \begin{bmatrix} X_{t_{-1}} \\ X_{t-1|t-1} \end{bmatrix} + \begin{bmatrix} V_{11} \\ KDV_{11} \end{bmatrix} \varepsilon_t \tag{2.24}$$

where K is the Kalman gain given by

$$K = PD'\,(DPD')^{-1} \tag{2.25}$$

$$P = (W_{11} + W_{12})\left(P - PD'\,(DPD')^{-1}\,DP\right)(W_{11} + W_{12})' + V_{11}\Sigma_{\varepsilon\varepsilon}V_{11}' \tag{2.26}$$

2.5. **Algorithm to solve model.**

(1) Find $\mathbf{g}'$ in full information solution (2.12) using undetermined coefficient method, as outlined in Campbell (1994).

(2) Find K and P by iterating on (2.26) starting from $P_0 = V_{11}\Sigma_{\varepsilon\varepsilon}V_{11}'$

(3) Plug K and $\mathbf{g}'$ into law of motion for system (2.24)

(4) Done.

The solution algorithm is somewhat simplified by the fact that all observable variables are either exogenous or predetermined. For a more general method that also work when contemporaneous endogenous variables are observed, see Svensson and Woodford (2004) and for an application to monetary policy, see Nimark (2008).

2.6. **Properties of the model.** Bomfim calibrates the model using standard parameters from the RBC literature for the structural parameters. He then calibrates the noise in the signals using what he calls "typical signal to noise ratios of economic indicators from the real world". Bomfim then reports results from two different exercises:

(1) What happens to the volatility of endogenous variables when the noise is reduced and agents use optimal filtering?

(2) What happens when noise is reduced if agents take announcements at face value?

The result of the first exercise is that the variance of all endogenous variables increase when noise is reduced, and the opposite happens when agents take the noisy indicators at face value (but Bomfim is not explicit about how this is computed). Parts of this results can be understood without resorting to simulations.

2.7. **The variance of consumption.** We can write consumption in the partial information model $c_t = \mathbf{g}'X_{t|t}$ as the obvious identity

$$\mathbf{g}'X_{t|t} = \mathbf{g}'X_{t|t}$$

and then add the full information consumption to both sides

$$\mathbf{g}'X_t + \mathbf{g}'X_{t|t} = \mathbf{g}'X_t + \mathbf{g}'X_{t|t}$$

Rearranging, we find that

$$\mathbf{g}'X_t = \mathbf{g}'\left(X_t - X_{t|t}\right) + \mathbf{g}'X_{t|t}$$

or that full information consumption is the sum of partial information consumption and a linear function of the state reconstruction error $\left(X_t - X_{t|t}\right)$. The variance of both sides have to be the same, and we know that from the optimality of the estimate $X_{t|t}$ that the error $\left(X_t - X_{t|t}\right)$ is orthogonal to $X_{t|t}$. This means that we can just sum up the variances on the right hand side without worrying about covariances to get

$$\mathbf{g}'E\left[X_tX_t'\right]\mathbf{g} = \mathbf{g}'E\left[\left(X_t - X_{t|t}\right)\left(X_t - X_{t|t}\right)'\right]\mathbf{g} + \mathbf{g}'E\left[X_{t|t}X_{t|t}'\right]\mathbf{g}$$

Since $\mathbf{g}'E\left[\left(X_t - X_{t|t}\right)\left(X_t - X_{t|t}\right)'\right]\mathbf{g}$ is a positive definite matrix, we know that $\mathbf{g}'E\left[X_tX_t'\right]\mathbf{g}$ $\geq \mathbf{g}'E\left[X_{t|t}X_{t|t}'\right]\mathbf{g}$, or that the variance of consumption under full information is larger than the variance of consumption under partial information.

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