TOPICS IN MACROECONOMICS: MODELLING INFORMATION, LEARNING AND EXPECTATIONS

LECTURE NOTES 5

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Private and public information

Most economic models involve some type of interaction between multiple agents where the payoff of one agent depends not only on the actions taken by him, but also on the actions taken by other agents. When agents' preferences and environment are identical and all share the same information, an individual agent can infer the actions that others will take by introspection, since all agents will choose the same action in equilibrium. If agents have access to private information, this is no longer possible since individual agents cannot know with certainty what other agents know and therefore also not know with certainty what actions they will take. It then becomes necessary for agents to form expectations about the actions of others. Additionally, to predict the behavior of agents that form expectations about the actions of others, one need to form expectations about other agents' expectations about the actions of others, and so on, leading to the so-called infinite regress of expectations.[1] The idea that agents observe different pieces of information has a lot of appeal and has been applied to a variety of settings, including general equilibrium models of the business cycle and asset pricing models.[2] However, as a consequence of the infinite regress problem one could characterize most existing models of private information and strategic interaction as efforts to avoid modelling higher order expectations explicitly, and instead find alternative

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[1] Townsend (1983) and Sargent (1991).

[2] Some examples are Townsend (1983), Sargent (1991), Woodford (2002), Lorenzoni (2005), Bacchetta and Van Wincoop (2005), Kasa, Walker and Whiteman (2006) and Cespa and Vives (2007).

representations where higher order expectations do not occur as state variables. Notable exceptions are Woodford (2002), Morris and Shin (2002) and Adam (forthcoming) who by restricting their attention to models of static decisions are able to analyze higher order expectations explicitly.

The paper by Morris and Shin (2002) is (together with Townsend 1983) one of the most often cited papers in this literature. In it, the authors show that agents tend to behave as if they put "too much" weight on public signals relative to private signals when there are strategic complimentarities in actions, that is, when agent's utility is decreaisng in the distanceof their own action from others' actions. This makes intutiive sense: The public signal conveys information about the information available to others and naturally becomes more important when the actions of other agents matter for an individual's optimal decision. We will derive the solution and some of the results from Morris and Shin's paper below. Before analyzing the economics of private and public information though, it is necessary to invest some time in a notational machinery as well as to define exactly what is meant by a higher order expectation.

1. Higher order expectations:Concepts and notation

There is a continuum of agents indexed by j. Agent j's first order expectation of the variable θ_t is agent j's best estimate of the value of the variable given his information set $I_t(j)$. We denote agent j's first order expectation of θ_t at time t

$$\theta_{t|t}^{(1)}(j) \equiv E\left[\theta_t \mid I_t(j)\right] \tag{1.1}$$

The average first order expectation is obtained by taking averages of (1.1) across agents

$$\theta_{t|t}^{(1)} \equiv \int E\left[\theta_t \mid I_t(j)\right] \quad dj \tag{1.2}$$

The average second order expectation is obtained by taking the average of agents' expectations of (1.2)

$$\theta_{t|t}^{(2)} \equiv \int E\left[\theta_{t|t}^{(1)} \mid I_t(j)\right] \; dj \tag{1.3}$$

The average contemporaneous second order expectation of θ_t thus is the average expectation at time t of the average expectation at time t of the value of θ_t. We can generalize this notation to the k^{th} order expectation of θ_t

$$\theta_{t|t}^{(k)} \equiv \int E\left[\theta_{t|t}^{(k-1)} \mid I_t(j)\right] \quad dj \tag{1.4}$$

Define the zero order expectation of θ_t as the actual value of the variable

$$\theta_t^{(0)} \equiv \theta_t \tag{1.5}$$

In general

$$\theta_{t|t}^{(k)} \neq \theta_{t|t}^{(k+l)} \tag{1.6}$$

for $l \neq 0$. We call a sequence of expectations, for instance from order zero to k, a *hierarchy* of expectations from order zero to k. Vectors consisting of a hierarchy of expectations are denoted

$$\theta_{t|t}^{(0:k)} = \left[\begin{array}{cccc} \theta_{t|t}^{(0)} & \theta_{t|t}^{(1)} & \cdots & \theta_{t|t}^{(k)} \end{array} \right]' \tag{1.7}$$

2. Rationality and expectations about others' expectations

In rational expectations models, (first order) expectations are pinned down by the structure of the model. That is, an agent's expectations should be the mathematical expectation of the variable in question, conditional on the information set available to the agent. The underlying assumption we make is thus that agents know the structure of the economy, that is, agents know the functional form and true parameter values of the model. Similarly, second order knowledge of rationality can be used to pin down second order expectations. That is, a rational agent's expectations can also be predicted, and treated as a random variable like any other. If an agent wants to form an expectation about another agents expectation, and knows that the other agent is rational, then second order expectation will be the rational expetation conditional on the expected infromation set of the other agent. A similar logic can be applied to third an higher order expectations.

2.0.1. *A simple example.* Consider the unobservable variable θ given by

$$\theta \sim N\left(0, \sigma_\theta^2\right) \tag{2.1}$$

Agents (indexed by j) observe a private noisy signal of θ given by

$$\begin{aligned} z(j) &= \theta + \eta(j) \\ \eta(j) &\sim N\left(0, \sigma_\eta^2\right) \forall j \end{aligned} \tag{2.2}$$

That is, all agents receive an equally precise signal of θ but agent j only observes his own signal $z(j)$. The optimal estimate of θ conditional on $z(j)$ is then given by

$$E\left[\theta \mid z(j)\right] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\eta^2} z(j) \tag{2.3}$$

$$= gz(j) \tag{2.4}$$

To find the average first order expectation, we just take averages, that is integrate over j to get

$$\begin{aligned} \theta^{(1)} &= \int E\left[\theta \mid z(j)\right] \; dj \\ &= g\theta + g\int \eta(j)\; dj \\ &= g\theta \end{aligned}$$

where the last identity follows from the fact that the idiosyncratic noise terms average out to zero, i.e. $\int \eta(j)\; dj = 0$. The average first order expectation is thus a linear function of the true state θ. Agent j's second order expectation, that is agent j's expectation of the average expectation of θ is then given by

$$\begin{aligned} E\left[\theta^{(1)} \mid z(j)\right] &= gE\left[\theta \mid z(j)\right] && (2.5) \\ &= ggz(s) && (2.6) \end{aligned}$$

Again taking averages across agents gives us the average second order expectation of θ

$$\begin{aligned} \theta^{(2)} &= \int E\left[\theta^{(1)} \mid z(j)\right] \ \ dj && (2.7) \\ &= g^2\theta + g^2 \int \eta(j)\ dj && (2.8) \\ &= g^2\theta && (2.9) \end{aligned}$$

We could conitnue this indefinitely using that the average k^{th} order expectation will be given by

$$\theta^{(k)} = g^k\theta \qquad (2.10)$$

3. Public Information and Strategic Interaction

In an influential paper in the AER from 2002, Stephen Morris and Hyun Shin demonstrated that in the combination of strategic complementarity and private information can make the impact of public signals disproportionately large and markets can appear to "overreact" to news. This result is derived in a setting with fully rational agents. Whenever anyone writes a paper about this topic, it is customary to refer to Keynes' "beauty contest" metaphor of stock markets. So here it is:

"It is not a case of choosing those [faces] which, to the best of one's judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees." Keynes, General Theory of Employment Interest and Money, 1936.

3.1. **Morris and Shin's model.** Utility of agent i is given by

$$U_i = -(1-r)(a_i-\theta)^2 - r\left(L_i - \overline{L}\right) \tag{3.1}$$

where a_i is the action taken by agent i and

$$L_i = \int (a_j - a_i)^2 dj \tag{3.2}$$

and

$$\overline{L} = \int L_j dj \tag{3.3}$$

Agent i's first order condition is

$$a_i = (1-r)E[\theta \mid I(i)] + rE[\overline{a} \mid I(i)] \tag{3.4}$$

where $\overline{a}$ is the average action, i.e. $\int a_i di$. Agents observe two signals of θ. The public signal y

$$\begin{aligned} y &= \theta + \eta \\ \eta &\sim N(0, \sigma_\eta^2) \end{aligned} \tag{3.5}$$

and the private signal x_i

$$\begin{aligned} x_i &= \theta + \varepsilon_i \\ \varepsilon_i &\sim N(0, \sigma_\varepsilon^2)\forall i \end{aligned} \tag{3.6}$$

The conditional first order expectation of θ will then be

$$\begin{aligned} E[\theta \mid x_i, y] &= \frac{\sigma_\eta^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}(x_i - y) + y & (3.7) \\ &= g(x_i - y) + y & (3.8) \end{aligned}$$

and the general expression for a k^{th} order expectation

$$\theta^{(k)} = g^k (\theta - y) + y \tag{3.9}$$

3.2. **Equilibrium.** Conjecture a solution of the form

$$a_i = \kappa x_i + (1 - \kappa) y \tag{3.10}$$

Substitute into FOC to get

$$\kappa x_i + (1 - \kappa) y = (1 - r) [g (x_i - y) + y] + r [\kappa (g (x_i - y) + y) + (1 - \kappa) y] \tag{3.11}$$

equate coefficients on x_i

$$\begin{aligned} \kappa &= (1 - r) g + r \kappa g & (3.12) \\ &= \frac{(1 - r) g}{1 - r g} & (3.13) \end{aligned}$$

We can check that the limits makes sense:

- When σ_η^2 (and g) tends to zero, κ tends to zero: When the public signal is perfectly accurate, the optimal action put zero weight on the private signal.
- When σ_η^2 tends to infinity (and g tends to 1), κ tends to one: When the public signal is infinitely noisy, the optimal action put zero weight on the public signal.
- When $0 < g < 1$ and r decreases and tends to 0 (i.e. decreasing utility from coordination), κ increases from it's minimum zero (at $r = 1$) towards it maximum g (at $r = 0$) which also makes sense: When no weight is put on coordination ($r = 0$), agents will simply choose to minimize the distance between their action and the fundamental θ, i.e. $r = 0 \implies a_i = E [\theta \mid x_i, y] = g x_1 + (1 - g) y$.

3.3. **Welfare.** Morris and Shin specifies the following social welfare function

$$\begin{aligned} W &\equiv \frac{1}{1-r}\int u_i di & (3.14)\\ &= -\int (a_i-\theta)^2\, di & (3.15)\end{aligned}$$

That is, social welfare is a function solely of the average squared distance between actions and fundamentals. We can substitute in our optimal stragety to get

$$\begin{aligned} E\left[W \mid \theta\right] &= -\int (\kappa x_i + (1-\kappa)y - \theta)^2\, di & (3.16)\\ &= -\int \left[\kappa\left(\theta+\varepsilon_i\right) + (1-\kappa)\left(\theta+\eta\right) - \theta\right]^2 di & (3.17)\end{aligned}$$

Simplify

$$\begin{aligned} E\left[W \mid \theta\right] &= -\int \left[\kappa\varepsilon_i + (1-\kappa)\,\eta\right]^2 & (3.18)\\ &= -\kappa^2\sigma_\varepsilon^2 - (1-\kappa)^2\,\sigma_\eta^2 \end{aligned}$$

Morris and Shin then shows that

$$\frac{\partial E\left[W \mid \theta\right]}{\partial \sigma_\eta^2} < 0 \tag{3.19}$$

if and only if

$$\frac{\sigma_\eta^2}{\sigma_\varepsilon^2} \leq \frac{1}{(2r-1)(1-r)} \tag{3.20}$$

That is, more noise in the public signal can be good for welfare if private noise is large relative to public noise.

4. THE DYNAMICS OF HIGHER ORDER EXPECTATIONS: A SIMPLE EXAMPLE

This section provides a simple example taken from Nimark (2008b) of how common knowledge of rationality can help pin down a law of motion for higher order expectsations in a dynamic setting. We use a simple set up with a continuum of agents that are estimating an unobservable persistent process. Agents are rational Bayesians in the sense that they form optimal estimates of the process given their information sets and update their estimates when new information arrives by applying Bayes' law through the Kalman filter. This is common knowledge, i.e. all agents know that all agents know, and so on, that all agents are rational Bayesians. Agents' sole concern is to estimate an unobservable process and the average of other agents' estimate of the same process. This set up is transparent (but void of any economic meaning) but the intuition carries over to more interesting settings where agents also observe endogenous signals.

4.1. **Estimating a persistent process.**

Agents are indexed by $j \in (0,1)$ and estimate the unobservable persistent process

$$\begin{aligned} \theta_t &= \rho\theta_{t-1} + v_t && (4.1) \\ v_t &\sim N\left(0, \sigma_v^2\right) \end{aligned}$$

In period t agent j observes the unbiased but noisy signal $s_t(j)$ of the true value of θ_t

$$\begin{aligned} s_t(j) &= \theta_t + \eta_t(j) && (4.2) \\ \eta_t(j) &\sim N\left(0, \sigma_\eta^2\right) \ \forall \ \ j && (4.3) \end{aligned}$$

Equations (4.1) and (4.2) form a state space system that can be estimated using the Kalman filter. Agent j's optimal estimate of θ_t in period t is given by the updating equation

$$\begin{aligned} \theta_{t|t}^{(1)}(j) &= (1 - g_1)\,\rho\theta_{t-1|t-1}^{(1)}(j) + g_1 s_t(j) && (4.4) \\ g_1 &= \frac{p}{p + \sigma_\eta^2} < 1 && (4.5) \\ p &= \sigma_v^2 + p\rho^2 - \frac{(p\rho)^2}{p + \sigma_e^2} && (4.6) \end{aligned}$$

The interpretation of (4.4) is the following. The current estimate $\theta_{t|t}^{(1)}$ is a weighted average of the prior $\rho\theta_{t-1|t-1}$ and the observation $s_t(j)$. Intuitively, less weight is put on a noisy observation so the Kalman gain g_1 is decreasing in the variance of the noise term η_t. The limit cases $\sigma_\eta^2 = 0$ and $\sigma_\eta^2 = 1$ implies that $g_1 = 1$ and $g_1 = 0$ respectively so that $\sigma_\eta^2 = 0$ implies that the signal is a perfect indicator of the underlying variable since $s_t(j) = \theta_t \ \forall\ t, j$

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4.2. **Higher order estimates.** In order to derive a law of motion for higher order estimates, take averages of the updating equation (4.4) across agents and combine it with the actual process (4.1) to get

$$\begin{bmatrix} \theta_t \\ \theta^{(1)}_{t|t} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ g_1\rho & (1-g_1)\rho \end{bmatrix} \begin{bmatrix} \theta_{t-1} \\ \theta^{(1)}_{t-1|t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ g_1 \end{bmatrix} v_t \tag{4.7}$$

The system (4.7) is the joint law of motion for the actual and average first order expectation of θ_t. Higher order estimates can be added recursively by recognizing that individual agents can form an estimate of the system (4.7) by using the Kalman filter. The relevant state space system is then the transition (state) equation (4.7) and the measurement equation (4.8)

$$s_t(j) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_t \\ \theta^{(1)}_{t|t} \end{bmatrix} + \eta_t(j) \tag{4.8}$$

which is just a reformulation of (4.2). Applying standard formulas for a multivariate Kalman filter yields the updating equation for agent j's estimate of the system (4.7)

$$\begin{aligned} \begin{bmatrix} \theta^{(1)}_{t|t}(j) \\ \theta^{(2)}_{t|t}(j) \end{bmatrix} &= \left(I - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}' \right) \begin{bmatrix} \rho & 0 \\ g_1\rho & (1-g_1)\rho \end{bmatrix} \begin{bmatrix} \theta^{(1)}_{t-1|t-1}(j) \\ \theta^{(2)}_{t-1|t-1}(j) \end{bmatrix} \\ &\quad + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}' \begin{bmatrix} \theta_t \\ \theta^{(1)}_{t|t} \end{bmatrix} + \eta_t(j) \right) \end{aligned} \tag{4.9}$$

where g_1 and g_2 are the elements of the Kalman gain matrix. Taking averages of (4.9) across agents and amending it to (4.1) yields the law of motion for estimates of θ_t from order zero

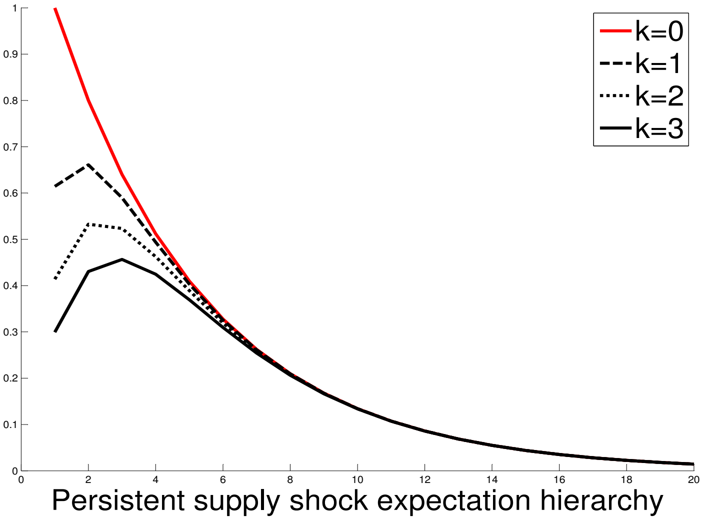

Figure 1

to two.

$$\begin{bmatrix} \theta_t^{(0)} \\ \theta_{t|t}^{(1)} \\ \theta_{t|t}^{(2)} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ g_1\rho & (1-g_1)\,\rho & 0 \\ g_2\rho & (g_1-g_2)\rho & (1-g_1)\,\rho \end{bmatrix} \begin{bmatrix} \theta_{t-1}^{(0)} \\ \theta_{t-1|t-1}^{(1)} \\ \theta_{t-1|t-1}^{(2)} \end{bmatrix} + \begin{bmatrix} 1 \\ g_1 \\ g_2 \end{bmatrix} v_t \tag{4.10}$$

We could in principle repeat this procedure to form ever larger state space systems, including higher and higher orders of estimates. However, the three dimensional system (4.10) is sufficient to illustrate how common knowledge of rationality can be used to construct a law of motion for higher order expectations. Figure 1 plots the impulse response of $\theta_{t|t}^{(0:3)}$ to a unit shock to θ_t.

5. Further reading

Most of these notes dealt with methodological concepts. For an application of these ideas to price setting in a macro context, see Woodford (2002) and Nimark (2008a). For more on asset pricing, see Bacchetta and van Wincoop (2006), Allen, Morris and Shin (2007) and Nimark (2008b) and various papers by Laura Veldkamp (NYU/Stern). Nimark (2008b) discusses methodological issues and provides conditions for when dynamic models with private information can be solved accurately. Below some additional papers of interest are also listed. The titles should be suggestive of their content.

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